

INSTITUTE OF THEORETICAL PHYSICS AND ASTRONOMY
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Julius Ruseckas

**MEASUREMENT MODELS FOR QUANTUM ZENO AND
ANTI-ZENO EFFECTS AND TIME DETERMINATION IN
QUANTUM MECHANICS**

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Scientific supervisor:

Prof. Habil. Dr. Bronislovas Kaulakys (Institute of Theoretical Physics and Astronomy of Vilnius University, physical sciences, physics – 02 P, mathematical and general theoretical physics, classical mechanics, quantum mechanics, relativity, gravitation, statistical physics, thermodynamics – 190 P)

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Julius Ruseckas

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ANTI-ZENONO EFEKTŲ APRAŠYMU IR LAIKO
KVANTINĖJE MECHANIKOJE APIBRĖŽIMUI**

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Mokslinis vadovas:

prof. habil. dr. Bronislovas Kaulakys (Vilniaus universiteto Teorinės fizikos ir astronomijos institutas, fiziniai mokslai, fizika – 02 P, matematinė ir bendroji teorinė fizika, klasikinė mechanika, kvantinė mechanika, reliatyvizmas, gravitacija, statistinė fizika, termodinamika – 190 P)

Contents

1	Introduction	7
2	List of publications	10
3	Measurements in quantum mechanics	11
4	Quantum Zeno and anti-Zeno effects	14
4.1	Introduction	14
4.2	Description of the measured system	15
4.3	Simple model of measurement and quantum Zeno effect	16
4.3.1	Model of the measurement	16
4.3.2	Measurement of the unperturbed system	18
4.3.3	Measurement of the perturbed system	19
4.3.4	The discrete spectrum	21
4.3.5	The decaying system	26
4.4	Free evolution and measurements	28
4.4.1	Jump probability	29
4.4.2	Example: two-level system	31
4.5	General expression for the quantum Zeno and anti-Zeno effects	32
4.5.1	Measurement of the unperturbed system	33
4.5.2	Measurement of the perturbed system	33
4.5.3	Free evolution and measurements	36
4.5.4	Simplification of the expression for the jump probability	37
4.6	Influence of the detector's temperature on the quantum Zeno effect	39
4.6.1	Model of the measurement	39
4.6.2	Solution of the master equation	41
4.6.3	Measurement of the unperturbed system	42
4.6.4	Measurement of the perturbed system	43
4.7	Quantum trajectory method for the quantum Zeno and anti-Zeno effects	45
4.7.1	Measurement of the unperturbed system	46
4.7.2	The detector	47
4.7.3	Stochastic methods	48
4.7.4	Stochastic simulation of the detector	50
4.7.5	Frequently measured perturbed two level system	51
4.7.6	Decaying system	57
4.7.7	Measurement of the decaying system	60
4.8	Summary	65

5	Weak measurements and time problem in quantum mechanics	67
5.1	Introduction	67
5.2	The concept of weak measurements	68
5.3	The time on condition that the system is in the given final state	70
5.3.1	The model of the time measurement	70
5.3.2	The dwell time	71
5.3.3	The time on condition that the system is in the given final state	72
5.3.4	Example: two-level system	74
5.4	Tunneling time	76
5.4.1	Determination of the tunneling time	77
5.4.2	The model of the time measurement	79
5.4.3	Measurement of the dwell time	80
5.4.4	Conditional probabilities and the tunneling time	80
5.4.5	Properties of the tunneling time	81
5.4.6	The reflection time	83
5.4.7	The asymptotic time	87
5.5	Weak measurement of arrival time	90
5.5.1	Arrival time in classical mechanics	91
5.5.2	Weak measurement of arrival time	93
5.5.3	Arrival time probability	95
5.6	Summary	97
6	Summary of the results and conclusions	99

1 Introduction

In quantum mechanics there are two kinds of dynamical rules which are used to describe the observed time dependence of quantum states. For closed systems the unitary evolution according to the Schrödinger equation is valid. On the other hand, measured system experiences “reduction” or “collapse of the wave function”. Such dualistic description has been a problem since early development of quantum mechanics. During recent years the measurement problem attracted much attention due to the advancement in experimental techniques. Nevertheless, the full understanding of quantum-mechanical measurements has not been achieved as yet. The collapse of the wave function refers only to an ideal measurement, which is instantaneous and arbitrarily accurate. Real measurements are represented by the projection postulate only approximately.

When the unitary Schrödinger dynamics of a system is disturbed by the interaction with an environment, the most important effect is creation of the entanglement with an environment and resulting disturbance of the phase relations between the states of the system. This process is called decoherence. Without coupling to the environment the system displays a certain time dependence, governed by its own Hamiltonian. The motion of a system may become frozen by repeated measurements, even if these are performed ideally. This phenomenon is called the quantum Zeno effect.

The simplest analysis of the quantum Zeno effect does not take into account the actual mechanism of the measurement process involved. It is based on an alternating sequence of unitary evolution and a collapse of the wave function. Later it was realized that the repeated measurements could not only slow down the quantum dynamics but the quantum process may be accelerated by frequent measurements as well. This effect is called the quantum anti-Zeno effect.

The quantum Zeno and anti-Zeno effects have attracted much attention. Although a great progress in the investigation of the quantum Zeno effect has been made, this effect is not completely understood as yet. In a more accurate analysis of the quantum Zeno effect the finite duration of the measurement becomes important. Therefore, the projection postulate is not sufficient to solve this problem. The measurement should be described more fully, including the detector and the effect of the interaction with the environment into the description.

Another interesting concept related to the measurements in quantum mechanics is the idea of weak measurements by Aharonov, Albert and Vaidman. They showed that even when the interaction with the quantum system is very weak and it only slightly disturbs the dynamics of the system, it is possible to obtain some data about the quantum system averaging over a large ensemble of identical systems. Weak measurements are in some aspects more similar to the measurements in classical mechanics and can be used to obtain interesting results about some questions that are trivial in classical mechanics but lack rigorous formulation in quantum mechanics.

One of such a question is the tunneling time problem. There have been many attempts

to define a physical time for the tunneling processes. This question is still a subject of much controversy, since numerous theories contradict each other in their predictions for “the tunneling time”. Some of these theories predict that the tunneling process takes place at a speed faster than the speed of light, whereas the others state that it should be subluminal. The results of the experiments are often ambiguous. Using the weak measurements it is possible to gain some insight into the tunneling time problem.

Another problem that has attracted much attention is the quantum description of the arrival time. The problems rise from the fact, that the arrival time of a particle to a definite spatial point is a classical concept. In classical mechanics for the determination of a time the particle spends moving along a certain trajectory we have to measure the position of the particle at two different moments of time. In quantum mechanics this procedure does not work. Asking about the time in quantum mechanics one needs to define the procedure of the measurement.

The main goals of the research work

- To show that quantum Zeno and anti-Zeno effects can be understood as a consequence of the unitary Schrödinger dynamics and does not require a collapse of the wave function.
- To investigate simple models of measurement demonstrating the quantum Zeno and anti-Zeno effects.
- To derive a general equation for the probability of the jump during the measurement without using the collapse of the wave function.
- To compare the decay rate of the measured system given by the derived equation with the one obtained using more complete models of measurement that include into description the environment of the detector.
- To use the concept of weak measurements for the investigation of the tunneling time problem.
- Using weak measurements to investigate the problem of arrival time in quantum mechanics.

The main conclusions

1. The quantum Zeno and anti-Zeno effects can be described without collapse of the wave function.
2. Frequent real measurements of the quantum system can cause inhibition as well as acceleration of the evolution of the system.
3. The general expression for the jump probability during the measurement, first obtained by Kofman and Kurizki can be derived using only the assumptions of the

non-demolition measurement and the Markovian approximation for the quantum dynamics of the detector.

4. The general expression for the decay rate of a frequently measured system gives a good agreement with the numerical simulation of the measured two level system and for the decaying one, showing the quantum Zeno and anti-Zeno effects.
5. The expression is introduced for the duration when the observable has a certain value subject to a condition that the system is found in a given final state. It is shown that this duration has a definite value only when the commutativity condition is fulfilled. In the opposite case two characteristic durations can be introduced which can be combined into one complex quantity.
6. The tunneling time, defined using the concept of weak measurements, has no definite value.
7. The density of one sided arrivals in quantum mechanics can be defined using weak measurements. In analogy with the complex tunneling time, the complex arrival time is introduced. It is shown that the proposed approach imposes an inherent limitation on the resolution of the arrival time.

Approbation of the results

The main results of the research described in this dissertation have been published in 9 scientific papers.

Personal contribution of the author

The author of the thesis has performed most of the analytical derivations of the equations as well as numerical calculations.

2 List of publications

1. J. Ruseckas, *Possibility of tunneling time determination*, Phys. Rev. A **63** (5), 052107 (2001).
2. J. Ruseckas and B. Kaulakys, *Time problem in quantum mechanics and weak measurements*, Phys. Lett. A **287** (5–6), 297–303 (2001).
3. J. Ruseckas and B. Kaulakys, *Real measurements and quantum Zeno effect*, Phys. Rev. A **63** (6), 062103 (2001).
4. J. Ruseckas, *Influence of the finite duration of the measurement on the quantum Zeno effect*, Phys. Lett. A **291** (4–5), 185–189 (2001).
5. J. Ruseckas, *Influence of the detector's temperature on the quantum Zeno effect*, Phys. Rev. A, **66** (1), 012105 (2002).
6. J. Ruseckas and B. Kaulakys, *Weak measurement of arrival time*, Phys. Rev. A, **66** (5), 052106 (2002).
7. J. Ruseckas and B. Kaulakys, *General expression for the quantum Zeno and anti-Zeno effects*, Phys. Rev. A, **69** (3), 032104 (2004).
8. J. Ruseckas and B. Kaulakys, *Time problem in quantum mechanics and its analysis by the concept of weak measurement*, Lithuanian. J. Phys. **44** (3) 161–182 (2004).
9. J. Ruseckas and B. Kaulakys, *Quantum trajectory method for the quantum Zeno and anti-Zeno effects*, Phys. Rev. A **73** (5), 052101 (2006).

3 Measurements in quantum mechanics

Quantum mechanics has shown an ever increasing range of applicability, making it more and more evident that the formalism describes some general properties of Nature. Despite this success of quantum theory, there is still no consensus about its interpretation. The main problems center around the notions of “observation” and “measurement”. The problem of the “classical limit” is at the heart of the interpretation problem. Most textbooks suggest that classical mechanics is in some sense contained in quantum mechanics as a special case. These standard arguments are insufficient for several reasons. It remains unexplained why macro-objects come only in narrow wave packets, even though the superposition principle allows far more nonclassical states. Measurement-like processes would necessarily produce nonclassical macroscopic states as a consequence of the unitary Schrödinger dynamics. An example is the famous Schrödinger cat [1–3].

It is now being realized that the assumption of a closed macroscopic system is not justified. Objects we usually call “macroscopic” are interacting with their environment in such a strong manner that they can never be considered as isolated.

According to a universal Schrödinger equation, quantum correlations with the environment are permanently created with a great efficiency for all macroscopic systems. Phase relations between the system states during the interaction with the environment are transferred to the phase relations between combined system and environment states, leading to decoherence. Decoherence is crucial in the measurement process. Measurement-like interactions cause a strong quantum entanglement of macroscopic objects with their environment. Since the state of the environment is generally inaccessible to the observer, the accompanying delocalization of phases then effectively destroys superpositions between macroscopically different states, so that the object appears to be in one or other of those states.

Since a system in contact with an environment is generally in an entangled state with the latter it does not possess a pure state for itself. For the purpose of observations concerning only the degrees of freedom of the system one can form a reduced density matrix by averaging over the states of the environment. By performing experiments including also the environment one could, in principle, observe quantum correlations between the system and the environment which cannot be described by the reduced density matrix. The Hamiltonian dynamics of the total system usually induces a non-unitary dynamics for the reduced density matrix of the system itself. The classical properties emerge in those physical situations in which the system is strongly entangled with the environment that cannot be controlled and where the reduced density matrix of the former evolves toward a decohered form with respect to certain properties. Then, while the system remains entangled with the environment, all observations at our disposal are compatible with classical statistics concerning these properties [4].

Measurements are usually described by means of observables, formally represented by Hermitean operators. Physical states are assumed to vary in time in accordance with a

dynamical law. In contrast, a measurement device is usually defined regardless of time. If the system is in a state $|\alpha\rangle$, any basis $\{|n\rangle\}$ in Hilbert space defines formal probabilities $p_n = |\langle n|\alpha\rangle|^2$ to find the system in a state $|n\rangle$. However, only a basis consisting of states that are not immediately destroyed by decoherence defines “realizable observables”.

The usual description of measurements is phenomenological. However, measurements should be described dynamically as interactions between the measured system and the measurement device. As discussed by von Neumann [5], this interaction must be diagonal with respect to the measurement basis. It should act on the quantum state of the device in such a way that the “pointer” moves into a position appropriate for being read, $|n\rangle|\Phi_0\rangle \rightarrow |n\rangle|\Phi_n\rangle$, where $|\Phi_0\rangle$ is the initial state of the measurement device and $|\Phi_n\rangle$ is the state corresponding to the measurement result n . The states $|\Phi_n\rangle$, representing different pointer positions, must approximately be mutually orthogonal.

In many situations the back-reaction of the environment on the considered system is negligibly small if the system is in a certain state. This is a typical situation of a measurement-like process. If the interaction Hamiltonian is of von Neumann’s form

$$\hat{H}_{\text{int}} = \sum_n |n\rangle\langle n| \otimes \hat{A}_n, \quad (3.1)$$

where \hat{A}_n are operators acting only in the Hilbert space of the environment, an eigenstate $|n\rangle$ of the “observable” measured by this interaction will not be changed, while the environment acquires information about the state $|n\rangle$ in the sense that its state changes in an n -dependent way. States which do not change or change minimally during the interaction with the environment are called “pointer states” [4, 6–8].

Because of dynamical superposition principle, an initial superposition $\sum c_n|n\rangle$ does not lead to definite pointer positions. If decoherence is neglected, one obtains their entangled superposition $\sum c_n|n\rangle|\Phi_n\rangle$. This dilemma represents the “quantum measurement problem”. Von Neumann’s interaction is nonetheless regarded as the first step of a measurement. Yet, a collapse seems still to be required — now in the measurement device rather than in the microscopic system. Because of the entanglement between the system and the apparatus, it would then affect the total system.

The apparatus itself, since macroscopic, will interact strongly with its environment. By the same mechanism, correlations are then delocalized again, leading to a diagonal density matrix for the system and the apparatus. After establishing the system-apparatus correlations, information about the measurement result is rapidly transferred to the environment E ,

$$\left(\sum_n c_n|n\rangle|\Phi_n\rangle \right) |E\rangle \rightarrow \sum_n c_n|n\rangle|\Phi_n\rangle|E_n\rangle$$

with $\langle E_m|E_n\rangle \approx 0$. Again, the interaction coupling the apparatus to its environment is assumed to have the form (3.1) thereby defining the “pointer states” $|\Phi_n\rangle$ dynamically.

Therefore, the description of the measurement can have various levels of complexity [4]:

1. The simplest description is obtained when we assume a collapse after each complete measurement. A sequence of measurements with a collapse of the wave function after every measurement event leads to a stochastic history of system states. The coarse grained system’s dynamics can be described by a rate equation for the density

matrix. One can test whether the collapse has occurred by coherently recombining the split components of the wave function.

2. The next level of description includes the measuring device. The resulting wave function is a superposition of all branches corresponding to possible measurement results. Coherence between different “histories” is fully preserved in this state. Inspecting the detector would reveal a superposition of different results. The density matrix of the measuring apparatus is not diagonal in the pointer states, hence it does not even describe an improper mixture of measurement results. Furthermore, arbitrary superpositions of macroscopically different pointer readings could be produced by appropriate manipulations.
3. For a full unitary description the environment has to be included. Only after establishing quantum correlations with the environment, the “pointer basis” is well defined, since phase relations between different states of the detector corresponding to the different measurement results are delocalized. The density matrix describing the apparatus now represents an improper mixture of measurement results. No manipulation of system or apparatus states can recover coherence, if the delocalization of phases into the environment is truly irreversible.

However, decoherence does not provide by itself a solution to interpretational problems, in particular that of measurement. We still need to explain the selection of one definite result out of the improper mixture and the corresponding fate of the measured system. A macroscopic measurement device is always assumed to decohere into its macroscopic pointer states. However, environment-induced decoherence by itself does not yet solve the problem, since the “pointer states” $|\Phi_n\rangle$ may be assumed to include the total environment. A way out of this dilemma with quantum mechanical concepts requires one of two possibilities: a modification of the Schrödinger equation that explicitly describes a collapse or an Everett type interpretation, in which all measurement outcomes are assumed to exist in one formal superposition.

Although incomplete, models of measurements with various levels of complexity are useful in investigation of the problems where the simplest description using wave function collapse is insufficient. In this thesis various models of the measurement are used to consider quantum Zeno and anti-Zeno effects and the problems related with time in quantum mechanics.

4 Quantum Zeno and anti-Zeno effects

4.1 Introduction

Theory of measurements has a special status in quantum mechanics. Unlike classical mechanics, in quantum mechanics it cannot be assumed that the effect of the measurement on the system can be made arbitrarily small. It is necessary to supplement quantum theory with additional postulates, describing the measurement. One of such additional postulate is von Neumann's state reduction (or projection) postulate [5]. The essential peculiarity of this postulate is its nonunitary character. However, this postulate refers only to an ideal measurement, which is instantaneous and arbitrarily accurate. Real measurements are described by the projection postulate only roughly.

The important consequence of von Neumann's projection postulate is the quantum Zeno effect. In quantum mechanics short-time behavior of nondecay probability of unstable particle is not exponential but quadratic [9]. This deviation from the exponential decay has been observed by Wilkinson *et al.* [10]. In 1977, Misra and Sudarshan [11] showed that this behavior when combined with the quantum theory of measurement, based on the assumption of the collapse of the wave function, led to a very surprising conclusion: frequent observations slowed down the decay. An unstable particle would never decay when continuously observed. Misra and Sudarshan have called this effect the quantum Zeno paradox or effect. The effect is called so in allusion to the paradox stated by Greek philosopher Zeno (or Zenon) of Elea. The very first analysis does not take into account the actual mechanism of the measurement process involved, but it is based on an alternating sequence of unitary evolution and a collapse of the wave function.

Cook [12] suggested an experiment on the quantum Zeno effect that was realized by Itano *et al.* [13]. In this experiment a repeatedly measured two-level system undergoing Rabi oscillations has been used. The outcome of this experiment has also been explained without the collapse hypothesis [14–16]. Stochastic simulations of the quantum Zeno effect experiment were performed in Ref. [17]. Recently, an experiment similar to Ref. [13] has been performed by Balzer *et al.* [18]. The quantum Zeno effect has been considered for tunneling from a potential well into the continuum [19], as well as for photoionization [20]. The quantum Zeno and anti-Zeno effects have been experimentally observed in an atomic tunneling process in Ref. [21].

Later it was realized that the repeated measurements could not only slow the quantum dynamics but the quantum process may be accelerated by frequent measurements as well [20, 22–26]. This effect was called a quantum anti-Zeno effect by Kaulakys and Gontis [22], who argued that frequent disturbances may destroy quantum localization effect in chaotic systems. An effect, analogous to the quantum anti-Zeno effect has been obtained in a computational study involving barrier penetration, too [27]. Recently, an analysis of the acceleration of a chemical reaction due to the quantum anti-Zeno effect

has been presented in Ref. [28].

A simple interpretation of quantum Zeno and anti-Zeno effects was given in Ref. [26]. Using projection postulate the universal formula describing both quantum Zeno and anti-Zeno effects was obtained. According to Ref. [26], the decay rate is determined by the convolution of two functions: the measurement-induced spectral broadening and the spectrum of the reservoir to which the decaying state is coupled.

The states of the frequently measured system need not be frozen: in the general situation coherent evolution of the system can take place in dynamically generated quantum Zeno subspaces [29]. The projective measurements used in the description of the quantum Zeno effect can be replaced by another quantum system interacting strongly with the principal system [30–33].

Quantum Zeno effect can have practical significance in quantum computing. The use of the quantum Zeno effect for correcting errors in quantum computers was first suggested by Zurek [34]. A number of quantum codes utilising the error prevention that occurs in the Zeno limit have been proposed [35–39].

In the analysis of the quantum Zeno effect the finite duration of the measurement becomes important, therefore, the projection postulate is not sufficient to solve this problem. The complete analysis of the Zeno effect requires a more precise model of measurement than the projection postulate. The measurement should be described more fully, including in the description the detector. The basic ideas of a quantum measurement process were theoretically expounded in Refs. [4,6–8,40–42] on the assumption of environmentally induced decoherence or superselection. Therefore, in order to correctly describe the measurement process one should include in the description the interaction of the detector with the environment. Regardless of that, simpler models of measurement also can correctly describe some aspects of quantum Zeno and anti-Zeno effects.

This Chapter is organized as follows. We begin from simple model of measurement in Sec. 4.3 In Sec. 4.5 we analyze the quantum Zeno and anti-Zeno effects without using any particular measurement model and making only few assumptions. We obtain a more general expression for the jump probability during the measurement. Concrete model of the environment of the detector is used in Sec. 4.6. Another model of the detector is considered in Sec. 4.7. The influence of the environment is taken into account using quantum trajectory method.

4.2 Description of the measured system

We consider a system that consists of two parts. The first part of the system has the discrete energy spectrum. The Hamiltonian of this part is \hat{H}_0 . The other part of the system is represented by Hamiltonian \hat{H}_1 . The Hamiltonian \hat{H}_1 commutes with \hat{H}_0 . In a particular case the second part can be absent, i.e. \hat{H}_1 can be zero. The operator $\hat{V}(t)$ causes the jumps between different energy levels of \hat{H}_0 . Therefore, the full Hamiltonian of the system is of the form $\hat{H}_S = \hat{H}_0 + \hat{H}_1 + \hat{V}(t)$. The example of such a system is an atom with the Hamiltonian \hat{H}_0 interacting with the electromagnetic field, represented by \hat{H}_1 , while the interaction between the atom and the field is $\hat{V}(t)$.

We will measure in which eigenstate of the Hamiltonian \hat{H}_0 the system is. The measurement is performed by coupling the system with the detector. The full Hamiltonian of

the system and the detector equals to

$$\hat{H} = \hat{H}_S + \hat{H}_D + \hat{H}_I, \quad (4.1)$$

where \hat{H}_D is the Hamiltonian of the detector and \hat{H}_I represents the interaction between the detector and the measured system, described by the Hamiltonian \hat{H}_0 . We can choose the basis $|n\alpha\rangle = |n\rangle \otimes |\alpha\rangle$ common for the operators \hat{H}_0 and \hat{H}_1 ,

$$\hat{H}_0|n\rangle = E_n|n\rangle, \quad (4.2)$$

$$\hat{H}_1|\alpha\rangle = E_\alpha|\alpha\rangle, \quad (4.3)$$

where n numbers the eigenvalues of the Hamiltonian \hat{H}_0 and α represents the remaining quantum numbers.

4.3 Simple model of measurement and quantum Zeno effect

The complete analysis of the Zeno effect requires a more precise model of measurement than the projection postulate. The purpose of this section is to consider a simple model of the measurement. The model describes a measurement of the finite duration and finite accuracy. Although the used model does not describe the irreversible process, it leads, however, to the correct correlation between the states of the measured system and the measuring apparatus.

Due to the finite duration of the measurement it is impossible to consider infinitely frequent measurements, as in Ref. [11]. The highest frequency of the measurements is achieved when the measurements are performed one after another, without the period of the measurement-free evolution between two successive measurements. In our model we consider such a sequence of the measurements. Our goal is to check whether this sequence of the measurements can change the evolution of the system and to verify the predictions of the quantum Zeno effect.

4.3.1 Model of the measurement

The measured system is described in section 4.2. We choose the operator \hat{H}_I representing the interaction between the detector and the measured system in the form

$$\hat{H}_I = \lambda\hat{q}\hat{H}_0 \quad (4.4)$$

where \hat{q} is the operator acting in the Hilbert space of the detector and the parameter λ describes the strength of the interaction. This system—detector interaction is that considered by von Neumann [5] and in Refs. [43–47]. In order to obtain a sensible measurement, the parameter λ must be large. We require a continuous spectrum of operator \hat{q} . For simplicity, we can consider the quantity q as the coordinate of the detector.

The measurement begins at time moment t_0 . At the beginning of the interaction with the detector, the detector is in the pure state $|\Phi\rangle$. The full density matrix of the system

and detector is $\hat{\rho}(t_0) = \hat{\rho}_S(t_0) \otimes |\Phi\rangle\langle\Phi|$ where $\hat{\rho}_S(t_0)$ is the density matrix of the system. The duration of the measurement is τ . After the measurement the density matrix of the system is $\hat{\rho}_S(\tau + t_0) = \text{Tr}_D \left\{ \hat{U}(\tau + t_0) (\hat{\rho}_S(t_0) \otimes |\Phi\rangle\langle\Phi|) \hat{U}^\dagger(\tau + t_0) \right\}$ and the density matrix of the detector is $\hat{\rho}_D(\tau + t_0) = \text{Tr}_S \left\{ \hat{U}(\tau + t_0) (\hat{\rho}_S(t_0) \otimes |\Phi\rangle\langle\Phi|) \hat{U}^\dagger(\tau + t_0) \right\}$ where $\hat{U}(t)$ is the evolution operator of the system and detector, obeying the equation

$$i\hbar \frac{\partial}{\partial t} \hat{U}(t) = \hat{H}(t) \hat{U}(t) \quad (4.5)$$

with the initial condition $\hat{U}(t_0) = 1$.

Since the initial density matrix is chosen in a factorizable form, the density matrix of the system after the interaction depends linearly on the density matrix of the system before the interaction. We can represent this fact by the equality

$$\hat{\rho}_S(\tau + t_0) = S(\tau, t_0) \hat{\rho}_S(t_0) \quad (4.6)$$

where $S(\tau, t_0)$ is the superoperator acting on the density matrices of the system. If the vectors $|n\rangle$ form the complete basis in the Hilbert space of the system we can rewrite Eq. (4.6) in the form

$$\rho_S(\tau + t_0)_{pr} = S(\tau, t_0)_{pr}^{nm} \rho_S(t_0)_{nm} \quad (4.7)$$

where the sum over the repeating indices is supposed. The matrix elements of the superoperator are

$$S(\tau, t_0)_{pr}^{nm} = \text{Tr}_D \left\{ \langle p | \hat{U}(\tau + t_0) (|n\rangle\langle m| \otimes |\Phi\rangle\langle\Phi|) \hat{U}^\dagger(\tau + t_0) |r\rangle \right\}. \quad (4.8)$$

Due to the finite duration of the measurement it is impossible to realize the infinitely frequent measurements. The highest frequency of the measurements is achieved when the measurements are performed one after another without the period of the measurement-free evolution between two successive measurements. Therefore, we model a continuous measurement by the subsequent measurements of the finite duration and finite accuracy. After N measurements the density matrix of the system is

$$\hat{\rho}_S(N\tau) = S(\tau, (N-1)\tau) \dots S(\tau, \tau) S(\tau, 0) \hat{\rho}_S(0). \quad (4.9)$$

Further, for simplicity we will neglect the Hamiltonian of the detector. After this assumption the evolution operator is equal to $\hat{U}(t, 1 + \lambda\hat{q})$ where the operator $\hat{U}(t, \xi)$ obeys the equation

$$i\hbar \frac{\partial}{\partial t} \hat{U}(t, \xi) = \left(\xi \hat{H}_0 + \hat{H}_1 + \hat{V}(t + t_0) \right) \hat{U}(t, \xi) \quad (4.10)$$

with the initial condition $\hat{U}(t_0, \xi) = 1$. Then the superoperator $S(\tau, t_0)$ is

$$S(\tau, t_0)_{pr}^{nm} = \int |\langle q | \Phi \rangle|^2 \langle p | \hat{U}(\tau + t_0, 1 + \lambda q) |n\rangle \langle m | \hat{U}^\dagger(\tau + t_0, 1 + \lambda q) |r\rangle dq. \quad (4.11)$$

4.3.2 Measurement of the unperturbed system

In order to estimate the necessary duration of the single measurement it is convenient to consider the case when the operator $\hat{V} = 0$. In such a case the description of the evolution is simpler. The measurement of this kind occurs also when the influence of the perturbation operator \hat{V} is small in comparison with the interaction between the system and the detector and, therefore, the operator \hat{V} can be neglected. Since the Hamiltonian of the system does not depend on t we will omit the parameter t_0 in this section. From Eq. (4.11) we obtain the superoperator $S(\tau)$ in the basis $|n\alpha\rangle$

$$S(\tau)_{p\alpha_3, r\alpha_4}^{n\alpha_1, m\alpha_2} = \delta_{np}\delta_{mr}\delta(\alpha_1, \alpha_3)\delta(\alpha_2, \alpha_4) \exp(i\omega_{m\alpha_2, n\alpha_1}\tau) \times \int |\langle q|\Phi\rangle|^2 \exp(i\lambda\omega_{mn}\tau q) dq \quad (4.12)$$

where

$$\omega_{mn} = \frac{1}{\hbar}(E_m - E_n), \quad (4.13)$$

$$\omega_{m\alpha_2, n\alpha_1} = \omega_{mn} + \frac{1}{\hbar}(E_{\alpha_2} - E_{\alpha_1}) \quad (4.14)$$

and $\delta(\cdot, \cdot)$ represent the Kronecker's delta in a discrete case and the Dirac's delta in a continuous case. Eq. (4.12) can be rewritten using the correlation function

$$F(\nu) = \langle \Phi | \exp(i\nu\hat{q}) | \Phi \rangle. \quad (4.15)$$

We can express this function as $F(\nu) = \int |\langle q|\Phi\rangle|^2 \exp(i\nu q) dq = \int \langle \Phi|p\rangle \langle p - \frac{\nu}{\hbar}|\Phi\rangle dp$. Since vector $|\Phi\rangle$ is normalized, the function $F(\nu)$ tends to zero when $|\nu|$ increases. There exists a constant C such that the correlation function $|F(\nu)|$ is small if the variable $|\nu| > C$.

Then the equation for the superoperator $S(\tau)$ is

$$S(\tau)_{p\alpha_3, r\alpha_4}^{n\alpha_1, m\alpha_2} = \delta_{np}\delta_{mr}\delta(\alpha_1, \alpha_3)\delta(\alpha_2, \alpha_4) \exp(i\omega_{m\alpha_2, n\alpha_1}\tau) F(\lambda\tau\omega_{mn}). \quad (4.16)$$

Using Eqs. (4.7) and (4.16) we find that after the measurement the non-diagonal elements of the density matrix of the system become small, since $F(\lambda\tau\omega_{mn})$ is small for $n \neq m$ when $\lambda\tau$ is large.

The density matrix of the detector is

$$\langle q|\hat{\rho}_D(\tau)|q_1\rangle = \langle q|\Phi\rangle\langle\Phi|q_1\rangle \text{Tr} \left\{ \hat{U}(\tau, 1 + \lambda q)\hat{\rho}_S(0)\hat{U}^\dagger(\tau, 1 + \lambda q_1) \right\}. \quad (4.17)$$

From Eqs. (4.10) and (4.17) we obtain

$$\langle q|\hat{\rho}_D(\tau)|q_1\rangle = \langle q|\Phi\rangle\langle\Phi|q_1\rangle \sum_n \exp(i\lambda\tau\omega_n(q_1 - q)) \sum_\alpha \langle n, \alpha|\hat{\rho}_S(0)|n, \alpha\rangle \quad (4.18)$$

where

$$\omega_n = \frac{1}{\hbar}E_n. \quad (4.19)$$

The probability that the system is in the energy level n may be expressed as

$$P(n) = \sum_\alpha \langle n, \alpha|\hat{\rho}_S(0)|n, \alpha\rangle. \quad (4.20)$$

Introducing the state vectors of the detector

$$|\Phi_E\rangle = \exp\left(-\frac{i}{\hbar}\lambda\tau E\hat{q}\right)|\Phi\rangle \quad (4.21)$$

we can express the density operator of the detector as

$$\hat{\rho}_D(\tau) = \sum_n |\Phi_{E_n}\rangle\langle\Phi_{E_n}|P(n). \quad (4.22)$$

The measurement is complete when the states $|\Phi_E\rangle$ are almost orthogonal. The different energies can be separated only when the overlap between the corresponding states $|\Phi_E\rangle$ is almost zero. The scalar product of the states $|\Phi_E\rangle$ with different energies E_1 and E_2 is

$$\langle\Phi_{E_1}|\Phi_{E_2}\rangle = F(\lambda\tau\omega_{12}). \quad (4.23)$$

The correlation function $|F(\nu)|$ is small when $|\nu| > C$. Therefore, we have the estimation for the error of the energy measurement ΔE as

$$\lambda\tau\Delta E \gtrsim \hbar C \quad (4.24)$$

and we obtain the expression for the necessary duration of the measurement

$$\tau \gtrsim \frac{\hbar}{\Lambda\Delta E} \quad (4.25)$$

where

$$\Lambda = \frac{\lambda}{C}. \quad (4.26)$$

Since in our model the measurements are performed immediately one after the other, from Eq. (4.25) it follows that the rate of measurements is proportional to the strength of the interaction λ between the system and the measuring device.

4.3.3 Measurement of the perturbed system

The operator $\hat{V}(t)$ represents the perturbation of the unperturbed Hamiltonian $\hat{H}_0 + \hat{H}_1$. We will take into account the influence of the operator \hat{V} by the perturbation method, assuming that the strength of the interaction λ between the system and detector is large.

The operator $\hat{V}(t)$ in the interaction picture is

$$\tilde{V}(t, t_0, \xi) = \exp\left(\frac{i}{\hbar}(\xi\hat{H}_0 + \hat{H}_1)t\right)\hat{V}(t + t_0)\exp\left(-\frac{i}{\hbar}(\xi\hat{H}_0 + \hat{H}_1)t\right). \quad (4.27)$$

In the second order approximation the evolution operator equals to

$$\begin{aligned} \hat{U}(\tau, t_0, \xi) \approx & \exp\left(-\frac{i}{\hbar}(\xi\hat{H}_0 + \hat{H}_1)\tau\right)\left\{1 + \frac{1}{i\hbar}\int_0^\tau \tilde{V}(t, t_0, \xi)dt \right. \\ & \left. - \frac{1}{\hbar^2}\int_0^\tau dt_1 \int_0^{t_1} dt_2 \tilde{V}(t_1, t_0, \xi)\tilde{V}(t_2, t_0, \xi)\right\}. \end{aligned} \quad (4.28)$$

We obtain the superoperator S in the second order approximation substituting the approximate expression for the evolution operator (4.28) into Eq. (4.11). Thus we have

$$S(\tau, t_0) = S^{(0)}(\tau) + S^{(1)}(\tau, t_0) + S^{(2)}(\tau, t_0) \quad (4.29)$$

where $S^{(0)}(\tau)$ is the superoperator of the unperturbed measurement given by Eq. (4.16), $S^{(1)}(\tau, t_0)$ is the first order correction,

$$\begin{aligned} S^{(1)}(\tau, t_0)_{p\alpha_3, r\alpha_4}^{n\alpha_1, m\alpha_2} &= \frac{1}{i\hbar} \delta_{rm} \delta(\alpha_4, \alpha_2) \exp(i\omega_{r\alpha_4, p\alpha_3} \tau) \int_0^\tau dt V_{p\alpha_3, n\alpha_1}(t + t_0) \\ &\quad \times \exp(i\omega_{p\alpha_3, n\alpha_1} t) F(\lambda(\omega_{rp} \tau + \omega_{pn} t)) \\ &\quad - \frac{1}{i\hbar} \delta_{pn} \delta(\alpha_3, \alpha_1) \exp(i\omega_{r\alpha_4, p\alpha_3} \tau) \int_0^\tau dt V_{m\alpha_2, r\alpha_4}(t + t_0) \\ &\quad \times \exp(i\omega_{m\alpha_2, r\alpha_4} t) F(\lambda(\omega_{rp} \tau + \omega_{mr} t)), \end{aligned} \quad (4.30)$$

and $S^{(2)}(\tau, t_0)$ is the second order correction,

$$\begin{aligned} S^{(2)}(\tau, t_0)_{p\alpha_3, r\alpha_4}^{n\alpha_1, m\alpha_2} &= \frac{1}{\hbar^2} \exp(i\omega_{r\alpha_4, p\alpha_3} \tau) \int_0^\tau dt_1 \int_0^{t_1} dt_2 V_{p\alpha_3, n\alpha_1}(t_1 + t_0) V_{m\alpha_2, r\alpha_4}(t_2 + t_0) \\ &\quad \times F(\lambda(\omega_{rp} \tau + \omega_{pn} t_1 + \omega_{mr} t_2)) \exp(i\omega_{p\alpha_3, n\alpha_1} t_1 + i\omega_{m\alpha_2, r\alpha_4} t_2) \\ &\quad - \frac{1}{\hbar^2} \delta_{rm} \delta(\alpha_4, \alpha_2) \exp(i\omega_{r\alpha_4, p\alpha_3} \tau) \sum_{s, \alpha} \\ &\quad \int_0^\tau dt_1 \int_0^{t_1} dt_2 V_{p\alpha_3, s\alpha}(t_1 + t_0) V_{s\alpha, n\alpha_1}(t_2 + t_0) \\ &\quad \times F(\lambda(\omega_{rp} \tau + \omega_{ps} t_1 + \omega_{sn} t_2)) \exp(i\omega_{p\alpha_3, s\alpha} t_1 + i\omega_{s\alpha, n\alpha_1} t_2) \\ &\quad - \frac{1}{\hbar^2} \delta_{pn} \delta(\alpha_3, \alpha_1) \exp(i\omega_{r\alpha_4, p\alpha_3} \tau) \sum_{s, \alpha} \\ &\quad \int_0^\tau dt_1 \int_0^{t_1} dt_2 V_{m\alpha_2, s\alpha}(t_2 + t_0) V_{s\alpha, r\alpha_4}(t_1 + t_0) \\ &\quad \times F(\lambda(\omega_{rp} \tau + \omega_{sr} t_1 + \omega_{ms} t_2)) \exp(i\omega_{s\alpha, r\alpha_4} t_1 + i\omega_{m\alpha_2, s\alpha} t_2), \end{aligned} \quad (4.31)$$

The probability of the jump from the level $|i\alpha\rangle$ to the level $|f\alpha_1\rangle$ during the measurement is $W(i\alpha \rightarrow f\alpha_1) = S(\tau, t_0)_{f\alpha_1, i\alpha}^{i\alpha, i\alpha}$. Using Eqs. (4.29), (4.30) and (4.31) we obtain

$$\begin{aligned} W(i\alpha \rightarrow f\alpha_1) &= \frac{1}{\hbar^2} \int_0^\tau dt_1 \int_0^{t_1} dt_2 F(\lambda\omega_{if}(t_2 - t_1)) V(t_1 + t_0)_{f\alpha_1, i\alpha} V(t_2 + t_0)_{i\alpha, f\alpha_1} \\ &\quad \times \exp(i\omega_{i\alpha, f\alpha_1}(t_2 - t_1)). \end{aligned} \quad (4.32)$$

The expression for the jump probability can be further simplified if the operator \hat{V} does not depend on t . We introduce the function

$$\Phi_{f\alpha_1, i\alpha}(t) = |V_{f\alpha_1, i\alpha}|^2 \exp\left(\frac{i}{\hbar}(E_1(f, \alpha_1) - E_1(i, \alpha))t\right). \quad (4.33)$$

Changing variables we can rewrite the jump probability as

$$W(i\alpha \rightarrow f\alpha_1) = \frac{2}{\hbar^2} \text{Re} \int_0^\tau F(\lambda\omega_{fi}t) \exp(i\omega_{fi}t)(\tau - t) \Phi_{f\alpha_1, i\alpha}(t) dt. \quad (4.34)$$

Introducing the Fourier transformation of $\Phi_{f\alpha_1, i\alpha}(t)$

$$G_{f\alpha_1, i\alpha}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{f\alpha_1, i\alpha}(t) \exp(-i\omega t) dt \quad (4.35)$$

and using Eq. (4.34) we obtain the equality

$$W(i\alpha \rightarrow f\alpha_1) = \frac{2\pi\tau}{\hbar^2} \int_{-\infty}^{\infty} G_{f\alpha_1, i\alpha}(\omega) P_{if}(\omega) d\omega \quad (4.36)$$

where

$$P_{if}(\omega) = \frac{1}{\pi} \operatorname{Re} \int_0^\tau F(\lambda\omega_{if}t) \left(1 - \frac{t}{\tau}\right) \exp(i(\omega - \omega_{if})t) dt. \quad (4.37)$$

From Eq. (4.37), using the equality $F(0) = 1$, we obtain

$$\int P_{if}(\omega) d\omega = 1. \quad (4.38)$$

The quantity G equals to

$$G_{f\alpha_1, i\alpha}(\omega) = \hbar |V_{f\alpha_1, i\alpha}|^2 \delta(E_1(f, \alpha_1) - E_1(i, \alpha) - \hbar\omega). \quad (4.39)$$

We see that the quantity $G(\omega)$ characterizes the perturbation.

4.3.4 The discrete spectrum

Let us consider the measurement effect on the system with the discrete spectrum. The Hamiltonian \hat{H}_0 of the system has a discrete spectrum, the operator $\hat{H}_1 = 0$, and the operator $\hat{V}(t)$ represents a perturbation resulting in the quantum jumps between the discrete states of the system \hat{H}_0 .

For the separation of the energy levels, the error in the measurement should be smaller than the distance between the nearest energy levels of the system. It follows from this requirement and Eq. (4.25) that the measurement time $\tau \gtrsim \frac{1}{\Lambda\omega_{\min}}$, where ω_{\min} is the smallest of the transition frequencies $|\omega_{if}|$.

When λ is large then $|F(\lambda x)|$ is not very small only in the region $|x| < \Lambda^{-1}$. We can estimate the probability of the jump to the other energy level during the measurement, replacing $F(\nu)$ by $2C\delta(\nu)$ in Eq. (4.32). Then from Eq. (4.32) we obtain

$$W(i\alpha \rightarrow f\alpha_1) \approx \frac{2}{\hbar^2 \Lambda |\omega_{if}|} \int_0^\tau |V_{i\alpha_1, f\alpha}(t + t_0)|^2 dt. \quad (4.40)$$

We see that the probability of the jump is proportional to Λ^{-1} . Consequently, for large Λ , i.e. for the strong interaction with the detector, the jump probability is small. This fact represents the quantum Zeno effect. However, due to the finiteness of the interaction strength the jump probability is not zero. After sufficiently large number of measurements the jump occurs. We can estimate the number of measurements N after which the system jumps into other energy levels from the equality $\frac{2\tau}{\hbar^2 \Lambda |\omega_{\min}|} |V_{\max}|^2 N \sim 1$ where $|V_{\max}|$ is

the largest matrix element of the perturbation operator V . This estimation allows us to introduce the characteristic time, during which the evolution of the system is inhibited

$$t_{\text{inh}} \equiv \tau N = \Lambda \frac{\hbar^2 |\omega_{\min}|}{2 |V_{\max}|^2}. \quad (4.41)$$

We call this duration the inhibition time (it is natural to call this duration the Zeno time, but this term has already different meaning).

The full probability of the jump from level $|i\alpha\rangle$ to other levels is $W(i\alpha) = \sum_{f,\alpha_1} W(i\alpha \rightarrow f\alpha_1)$. From Eq. (4.40) we obtain

$$W(i\alpha) = \frac{2}{\hbar^2 \Lambda} \sum_{f,\alpha_1} \frac{1}{|\omega_{if}|} \int_0^\tau |V_{f\alpha_1, i\alpha}(t + t_0)|^2 dt. \quad (4.42)$$

If the matrix elements of the perturbation V between different levels are of the same size then the jump probability increases linearly with the number of the energy levels. This behavior has been observed in Ref. [48].

Due to the unitarity of the operator $\hat{U}(t, \xi)$ it follows from Eq. (4.11) that the superoperator $S(\tau, t_0)$ obeys the equalities

$$\sum_{p,\alpha} S(\tau, t_0)_{p\alpha, p\alpha}^{n\alpha_1, m\alpha_2} = \delta_{nm} \delta_{\alpha_1, \alpha_2}, \quad (4.43)$$

$$\sum_{n,\alpha} S(\tau, t_0)_{p\alpha_1, r\alpha_2}^{n\alpha, n\alpha} = \delta_{pr} \delta_{\alpha_1, \alpha_2}. \quad (4.44)$$

If the system has a finite number of energy levels, the density matrix of the system is diagonal and all states are equally occupied (i.e., $\rho_{n\alpha_1, m\alpha_2}(t_0) = \frac{1}{K} \delta_{nm} \delta_{\alpha_1, \alpha_2}$ where K is the number of the energy levels) then from Eq. (4.44) it follows that $S(\tau, t_0)\rho(t_0) = \rho(t_0)$. Such a density matrix is the stable point of the map $\rho \rightarrow S\rho$. Therefore, we can expect that after a large number of measurements the density matrix of the system tends to this density matrix.

When Λ is large and the duration of the measurement is small, we can neglect the non-diagonal elements in the density matrix of the system, since they always are of order Λ^{-1} . Replacing $F(\nu)$ by $2C\delta(\nu)$ in Eqs. (4.29), (4.30) and (4.31) and neglecting the elements of the superoperator S that cause the arising of the non-diagonal elements of the density matrix, we can write the equation for the superoperator S as

$$S(\tau, t_0)_{p\alpha_3, r\alpha_4}^{n\alpha_1, m\alpha_2} \approx \delta_{pn} \delta(\alpha_3, \alpha_1) \delta_{rm} \delta(\alpha_4, \alpha_2) \delta_{pr} + \frac{1}{\Lambda} A(\tau, t_0)_{p, \alpha_3, \alpha_4}^{n, \alpha_1, \alpha_2} \delta_{pr} \delta_{nm} \quad (4.45)$$

where

$$\begin{aligned} A(\tau, t_0)_{p, \alpha_3, \alpha_4}^{n, \alpha_1, \alpha_2} &= \frac{2}{\hbar^2 |\omega_{np}|} \int_0^\tau V_{p\alpha_3, n\alpha_1}(t + t_0) V_{n\alpha_2, p\alpha_4}(t + t_0) dt \\ &- \delta_{pn} \delta(\alpha_4, \alpha_2) \sum_{s,\alpha} \frac{1}{\hbar^2 |\omega_{sn}|} \int_0^\tau V_{n\alpha_3, s\alpha}(t + t_0) V_{s\alpha, n\alpha_1}(t + t_0) dt \\ &- \delta_{pn} \delta(\alpha_3, \alpha_1) \sum_{s,\alpha} \frac{1}{\hbar^2 |\omega_{ns}|} \int_0^\tau V_{s\alpha, n\alpha_4}(t + t_0) V_{n\alpha_2, s\alpha}(t + t_0) dt \end{aligned} \quad (4.46)$$

Then for the diagonal elements of the density matrix we have $\rho(\tau + t_0) \approx \rho(t_0) + \frac{1}{\Lambda} A(\tau, t_0) \rho(t_0)$, or

$$\frac{d}{dt} \hat{\rho}(t) \approx \frac{1}{\Lambda \tau} A(\tau, t) \hat{\rho}(t). \quad (4.47)$$

If the perturbation V does not depend on t then it follows from Eq. (4.47) that the diagonal elements of the density matrix evolve exponentially.

Example: two-level system

As an example we will consider the evolution of the measured two-level system. The system is forced by the perturbation V which induces the jumps from one state to another. The Hamiltonian of this system is

$$\hat{H} = \hat{H}_0 + \hat{V} \quad (4.48)$$

where

$$\hat{H}_0 = \frac{\hbar\omega}{2} \hat{\sigma}_3, \quad (4.49)$$

$$\hat{V} = v \hat{\sigma}_+ + v^* \hat{\sigma}_-. \quad (4.50)$$

Here $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices and $\sigma_{\pm} = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$. The Hamiltonian \hat{H}_0 has two eigenfunctions $|0\rangle$ and $|1\rangle$ with the eigenvalues $-\hbar\frac{\omega}{2}$ and $\hbar\frac{\omega}{2}$ respectively. The evolution operator of the unmeasured system is

$$\hat{U}(t) = \cos\left(\frac{\Omega}{2}t\right) - \frac{2i}{\hbar\Omega} \hat{H} \sin\left(\frac{\Omega}{2}t\right) \quad (4.51)$$

where

$$\Omega = \sqrt{\omega^2 + 4\frac{|v|^2}{\hbar^2}}. \quad (4.52)$$

If the initial density matrix is $\rho(0) = |1\rangle\langle 1|$ then the evolution of the diagonal elements of the unmeasured system's density matrix is given by the equations

$$\rho_{11}(t) = \cos^2\left(\frac{\Omega}{2}t\right) + \left(\frac{\omega}{\Omega}\right)^2 \sin^2\left(\frac{\Omega}{2}t\right) \quad (4.53)$$

$$\rho_{00}(t) = \left(1 - \left(\frac{\omega}{\Omega}\right)^2\right) \sin^2\left(\frac{\Omega}{2}t\right). \quad (4.54)$$

Let us consider now the dynamics of the measured system. The equations for the diagonal elements of the density matrix (Eq. (4.47)) for the system under consideration are

$$\frac{d}{dt} \rho_{11} \approx -\frac{1}{t_{\text{inh}}} (\rho_{11} - \rho_{00}), \quad (4.55)$$

$$\frac{d}{dt} \rho_{00} \approx -\frac{1}{t_{\text{inh}}} (\rho_{00} - \rho_{11}). \quad (4.56)$$

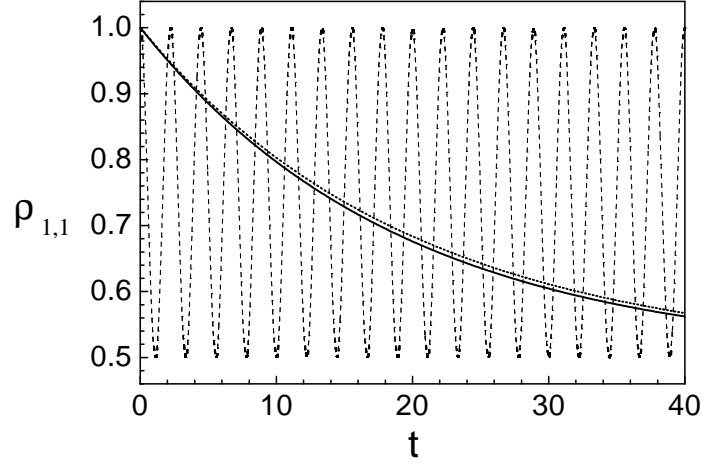


Figure 4.1: The occupation of the initial level 1 of the measured two-level system calculated according to Eqs. (4.7), (4.11), (4.51) and (4.61). The used parameters are $\hbar = 1$, $\sigma^2 = 1$, $\omega = 2$, $v = 1$. The strength of the measurement $\lambda = 50$ and the duration of the measurement $\tau = 0.1$. The exponential approximation (4.58) is shown as a dashed line. For comparison the occupation of the level 1 of the unmeasured system is also shown (dotted line).

where the inhibition time, according to Eq. (4.41), is

$$t_{\text{inh}} = \frac{\Lambda}{2\omega} \left| \frac{\hbar\omega}{v} \right|^2. \quad (4.57)$$

The solution of Eqs. (4.56) with the initial condition $\rho(0) = |1\rangle\langle 1|$ is

$$\rho_{11}(t) = \frac{1}{2} \left(1 + \exp\left(-\frac{2}{t_{\text{inh}}}t\right) \right), \quad (4.58)$$

$$\rho_{00}(t) = \frac{1}{2} \left(1 - \exp\left(-\frac{2}{t_{\text{inh}}}t\right) \right). \quad (4.59)$$

From Eq. (4.44) it follows that if the density matrix of the system is

$$\hat{\rho}_f = \frac{1}{2} (|0\rangle\langle 0| + |1\rangle\langle 1|), \quad (4.60)$$

then $S(\tau)\hat{\rho}_f = \hat{\rho}_f$. Hence, when the number of the measurements tends to infinity, the density matrix of the system approaches $\hat{\rho}_f$.

We have performed the numerical analysis of the dynamics of the measured two-level system (4.48)–(4.50) using Eqs. (4.7), (4.11) and (4.51) with the Gaussian correlation function (4.15)

$$F(\nu) = \exp\left(-\frac{\nu^2}{2\sigma^2}\right). \quad (4.61)$$

From the condition $\int_{-\infty}^{\infty} F(\nu)d\nu = 2C$ we have $C = \sigma\sqrt{\frac{\pi}{2}}$. The initial state of the system is $|1\rangle$. The matrix elements of the density matrix $\rho_{11}(t)$ and $\rho_{10}(t)$ are represented in

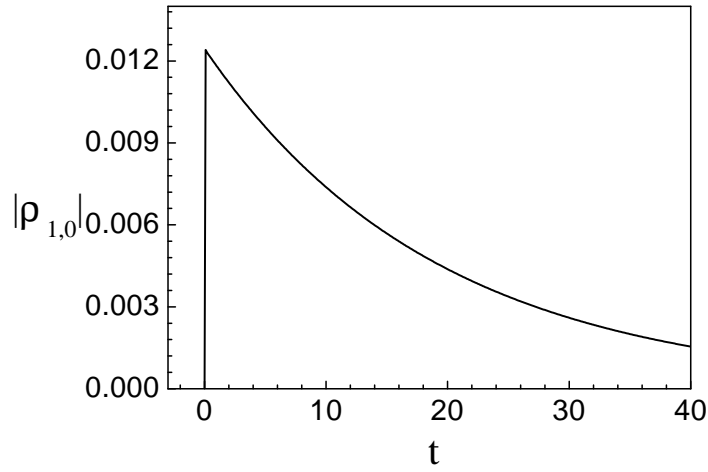


Figure 4.2: The non-diagonal element of the density matrix of the measured two-level system. Used parameters are the same as in Fig. 4.1

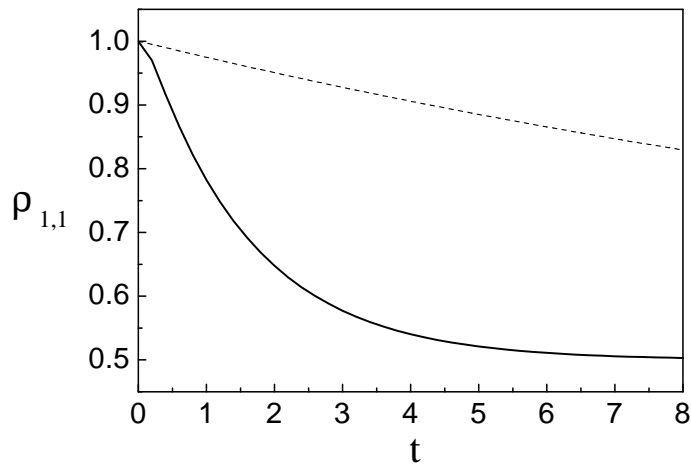


Figure 4.3: The occupation of the initial level 1 of the measured two-level system for different strengths of the measurement: $\lambda = 50, \tau = 0.1$ (dashed line) and $\lambda = 5, \tau = 0.2$ (solid line). Other parameters are the same as in Fig. 4.1

Fig. 4.1 and Fig. 4.2, respectively. In Fig. 4.1 the approximation (4.58) is also shown. This approach is close to the exact evolution. The matrix element $\rho_{11}(t)$ for two different values of λ is shown in Fig. 4.3. We see that for larger λ the evolution of the system is slower.

The influence of the repeated non-ideal measurements on the two level system driven by the periodic perturbation has also been considered in Refs. [48–51]. Similar results have been found: the occupation of the energy levels changes exponentially with time, approaching the limit $\frac{1}{2}$.

4.3.5 The decaying system

We consider the system which consists of two parts. We can treat the first part as an atom, and the second part as the field (reservoir). The energy spectrum of the atom is discrete and the spectrum of the field is continuous. The Hamiltonians of these parts are \hat{H}_0 and \hat{H}_1 respectively. There is the interaction between the atom and the field represented by the operator \hat{V} . So, the Hamiltonian of the system is

$$\hat{H}_S = \hat{H}_0 + \hat{H}_1 + \hat{V}. \quad (4.62)$$

When the measurement is not performed, such a system exhibits exponential decay, valid for the intermediate times. The decay rate is given according to the Fermi's Golden Rule

$$R(i\alpha_1 \rightarrow f\alpha_2) = \frac{2\pi}{\hbar} |V_{f\alpha_2, i\alpha_1}|^2 \rho(\hbar\omega_{if}) \quad (4.63)$$

where

$$\frac{1}{\hbar}(E_{\alpha_2} - E_{\alpha_1}) = \omega_{if} \quad (4.64)$$

and $\rho(E)$ is the density of the reservoir's states.

When the energy level of the atom is measured, we can use the perturbation theory, as it is in the discrete case.

The initial state of the field is a vacuum state $|0\rangle$ with energy $E_0 = 0$. Then the density matrix of the atom is $\hat{\rho}_0(\tau) = \text{Tr}_1\{\hat{\rho}(\tau)\} = \text{Tr}_1\{S(\tau)\hat{\rho}(0)\}$ or $\hat{\rho}_0(\tau) = S_{\text{eff}}(\tau)\hat{\rho}_0(0)$, where S_{eff} is an effective superoperator

$$S_{\text{eff}}(\tau)_{pr}^{nm} = \sum_{\alpha} S(\tau)_{p\alpha, r\alpha}^{n0, m0}. \quad (4.65)$$

When the states of the atom are weakly coupled to a broad band of states (continuum), the transitions back to the excited state of the atom can be neglected (i.e., we neglect the influence of emitted photons on the atom). Therefore, we can use the superoperator S_{ef} for determination of the evolution of the atom.

Since the states in the reservoir are very dense, one can replace the sum over α by an integral over E_{α}

$$\sum_{\alpha} \dots = \int dE_{\alpha} \rho(E_{\alpha}) \dots$$

where $\rho(E_{\alpha})$ is the density of the states in the reservoir.

The spectrum

The density matrix of the field is $\hat{\rho}_1(\tau) = \text{Tr}_0\{\hat{\rho}(\tau)\} = \text{Tr}_0\{S(\tau)\hat{\rho}(0)\}$. The diagonal elements of the field's density matrix give the spectrum. If the initial state of the atom is $|i\rangle$ then the distribution of the field's energy is $W(E_\alpha) = \rho_1(\tau)_{\alpha\alpha} = \sum_f S(\tau)_{f\alpha, f\alpha}^{i0, i0}$. From Eqs. (4.29), (4.30) and (4.31) we obtain

$$W(E_\alpha) = \sum_f \frac{2\pi}{\hbar^2} |V_{f\alpha, i0}|^2 \tau P_{if} \left(\frac{E_\alpha}{\hbar} \right) \quad (4.66)$$

where $P_{if}(\omega)$ is given by the equation (4.37). From Eq. (4.66) we see that $P(\omega)$ is the measurement-modified shape of the spectral line.

The integral in Eq. (4.37) is small when the exponent oscillates more rapidly than the function F . This condition is fulfilled when $\frac{E}{\hbar} - \omega_{if} \gtrsim \frac{\lambda\omega_{if}}{C}$. Consequently, the width of the spectral line is

$$\Delta E_{if} = \Lambda \hbar \omega_{if}. \quad (4.67)$$

The width of the spectral line is proportional to the strength of the measurement (this equation is obtained using the assumption that the strength of the interaction with the measuring device λ is large and, therefore, the natural width of the spectral line can be neglected). The broadening of the spectrum of the measured system is also reported in Ref. [24] for the case of an electron tunneling out of a quantum dot.

The decay rate

The probability of the jump from the state i to the state f is $W(i \rightarrow f; \tau) = S_{\text{eff}}(\tau)_{ff}^{ii}$. From Eqs. (4.65) it follows

$$W(i \rightarrow f; \tau) = \sum_\alpha W(i0, \rightarrow f\alpha, \tau) \quad (4.68)$$

Using Eq. (4.36) we obtain the equality

$$W(i \rightarrow f; \tau) = \frac{2\pi\tau}{\hbar^2} \int_{-\infty}^{+\infty} G_{fi}(\omega) P_{if}(\omega) d\omega. \quad (4.69)$$

where

$$G_{fi}(\omega) = \int \rho(E_\alpha) G_{f\alpha, i0}(\omega) dE_\alpha \quad (4.70)$$

The expression for $G(\omega)$ according to Eq. (4.39) is

$$G_{fi}(\omega) = \hbar \rho(\hbar\omega) |V_{fE_\alpha=\hbar\omega, i0}|^2. \quad (4.71)$$

The quantity $G(\omega)$ is the reservoir coupling spectrum.

The measurement-modified decay rate is $R(i \rightarrow f) = \frac{1}{\tau} W(i \rightarrow f; \tau)$. From Eq. (4.69) we have

$$R(i \rightarrow f) = \frac{2\pi}{\hbar^2} \int_{-\infty}^{\infty} G_{fi}(\omega) P_{if}(\omega) d\omega. \quad (4.72)$$

The equation (4.72) represents a universal result: the decay rate of the frequently measured decaying system is determined by the overlap of the reservoir coupling spectrum and the measurement-modified level width. This equation was derived by Kofman and Kurizki [26], assuming the ideal instantaneous projections. We show that Eq. (4.72) is valid for the more realistic model of the measurement, as well. An equation, similar to Eq. (4.72) has been obtained in Ref. [52], considering a destruction of the final decay state.

Depending on the reservoir spectrum $G(\omega)$ and the strength of the measurement the inhibition or acceleration of the decay can be obtained. If the interaction with the measuring device is weak and, consequently, the width of the spectral line is much smaller than the width of the reservoir spectrum, the decay rate equals the decay rate of the unmeasured system, given by the Fermi's Golden Rule (4.63). In the intermediate region, when the width of the spectral line is rather small compared with the distance between ω_{if} and the nearest maximum in the reservoir spectrum, the decay rate grows with increase of Λ . This results in the anti-Zeno effect.

If the width of the spectral line is much greater compared both with the width of the reservoir spectrum and the distance between ω_{if} and the centrum of the reservoir spectrum, the decay rate decreases when Λ increases. This results in the quantum Zeno effect. In such a case we can use the approximation

$$G_{fi}(\omega) \approx \hbar B_{fi} \delta(\omega - \omega_R). \quad (4.73)$$

where B_{fi} is defined by the equality $B_{fi} = \frac{1}{\hbar} \int G_{fi}(\omega) d\omega$ and ω_R is the centrum of $G(\omega)$. Then from Eq. (4.72) we obtain the decay rate $R(i \rightarrow f) \approx \frac{2\pi}{\hbar} B_{fi} P_{if}(\omega_{if})$. From Eq. (4.37), using the condition $\Lambda\tau|\omega_{if}| \gg 1$ and the equality $\int_{-\infty}^{\infty} F(\nu) d\nu = 2C$ we obtain

$$P_{if}(\omega_{if}) = \frac{1}{\pi\Lambda\omega_{if}}. \quad (4.74)$$

Therefore, the decay rate is equal to

$$R(i \rightarrow f) \approx \frac{2B_{fi}}{\Lambda\hbar\omega_{if}}. \quad (4.75)$$

The obtained decay rate is insensitive to the spectral shape of the reservoir and is inverse proportional to the measurement strength Λ .

4.4 Free evolution and measurements

In the analysis of the quantum Zeno effect the finite duration of the measurement becomes important. In Ref. [47] a simple model that allows us to take into account the finite duration and finite accuracy of the measurement has been developed. However, in Ref. [47] it has been analyzed the case when there are no free evolution between the measurements. In this section we obtain the corrections to the jump probability due to the finite duration of the measurement with the free evolution between the measurements.

The measured system is described in section 4.2. The measurement begins at time moment t_0 . At the beginning of the interaction with the detector, the detector is in

the pure state $|\Phi\rangle$. The full density matrix of the system and detector is $\hat{\rho}(t_0) = \hat{\rho}_S(t_0) \otimes |\Phi\rangle\langle\Phi|$ where $\hat{\rho}_S(t_0)$ is the density matrix of the system. The duration of the measurement is τ . After the measurement the density matrix of the system is $\hat{\rho}_S(\tau + t_0) = \text{Tr}_D\{\hat{U}_M(\tau, t_0)(\hat{\rho}_S(t_0) \otimes |\Phi\rangle\langle\Phi|)\hat{U}_M^\dagger(\tau, t_0)\}$ where $\hat{U}_M(t, t_0)$ is the evolution operator of the system and detector, obeying the equation

$$i\hbar\frac{\partial}{\partial t}\hat{U}_M(t, t_0) = \hat{H}(t + t_0)\hat{U}_M(t, t_0) \quad (4.76)$$

with the initial condition $\hat{U}_M(0, t_0) = 1$. Further, for simplicity we will neglect the Hamiltonian of the detector (as in Ref. [47]). Then the evolution operator \hat{U}_M obeys the equation

$$i\hbar\frac{\partial}{\partial t}\hat{U}_M(t, t_0) = \left((1 + \lambda\hat{q})\hat{H}_0 + \hat{H}_1 + \hat{V}(t + t_0)\right)\hat{U}_M(t, t_0). \quad (4.77)$$

After the measurement the system is left for the measurement-free evolution for time $T - \tau$. The density matrix becomes $\hat{\rho}_S(T + t_0) = \hat{U}_F(T - \tau, \tau + t_0)\hat{\rho}_S(\tau + t_0)\hat{U}_F^\dagger(T - \tau, \tau + t_0)$, where $\hat{U}_F(t, t_0)$ is the evolution operator of the system only, obeying the equation

$$i\hbar\frac{\partial}{\partial t}\hat{U}_F(t, t_0) = \hat{H}_S(t + t_0)\hat{U}_F(t, t_0) \quad (4.78)$$

with the initial condition $\hat{U}_F(0, t_0) = 1$.

The measurements of the duration τ with a subsequent free evolution for the time $T - \tau$ are repeated many times with the measurement period T . Such a process was considered by the Misra and Sudarshan [11] and realized in the experiments [13].

4.4.1 Jump probability

We will calculate the probability of the jump from the initial to the final state during the measurement and subsequent measurement-free evolution. The jumps are induced by the operator $\hat{V}(t)$ that represents the perturbation of the unperturbed Hamiltonian $\hat{H}_0 + \hat{H}_1$. We will take into account the influence of the operator \hat{V} by the perturbation method, assuming that the durations of the measurement τ and of the free evolution $T - \tau$ are small.

The operator $\hat{V}(t)$ in the interaction picture during the measurement is

$$\tilde{V}_M(t, t_0) = \hat{U}_M^{(0)}(t)\hat{V}(t + t_0)\hat{U}_M^{(0)}(t), \quad (4.79)$$

where $\hat{U}_M^{(0)}(t)$ is the evolution operator of the system and the detector (4.1) without the perturbation \hat{V}

$$\hat{U}_M^{(0)}(t) = \exp\left(-\frac{i}{\hbar}(\hat{H}_0 + \hat{H}_1 + \hat{H}_1)t\right). \quad (4.80)$$

The evolution operator $\hat{U}_M(\tau, t_0)$ in the second order approximation equals to

$$\begin{aligned} \hat{U}_M(\tau, t_0) \approx & \hat{U}_M^{(0)}(\tau) \left(1 + \frac{1}{i\hbar} \int_0^\tau dt \tilde{V}_M(t, t_0) \right. \\ & \left. - \frac{1}{\hbar^2} \int_0^\tau dt_1 \int_0^{t_1} dt_2 \tilde{V}_M(t_1, t_0) \tilde{V}_M(t_2, t_0) \right). \end{aligned} \quad (4.81)$$

The operator $\hat{V}(t)$ in the interaction picture during the free evolution is

$$\tilde{V}_F(t, t_0) = \hat{U}_F^{(0)}(t)\hat{V}(t + t_0)\hat{U}_F^{(0)}(t), \quad (4.82)$$

where $\hat{U}_F^{(0)}(t)$ is the evolution operator of the system without the perturbation \hat{V} , i.e.,

$$\hat{U}_F^{(0)}(t) = \exp\left(-\frac{i}{\hbar}(\hat{H}_0 + \hat{H}_1)t\right). \quad (4.83)$$

The evolution operator $\hat{U}_F(t, t_0)$ in the second order approximation equals to

$$\begin{aligned} \hat{U}_F(t, t_0) \approx & \hat{U}_F^{(0)}(t) \left(1 + \frac{1}{i\hbar} \int_0^t dt_1 \tilde{V}_F(t_1, t_0) \right. \\ & \left. - \frac{1}{\hbar^2} \int_0^t dt_1 \int_0^{t_1} dt_2 \tilde{V}_F(t_1, t_0) \tilde{V}_F(t_2, t_0) \right). \end{aligned} \quad (4.84)$$

The probability of the jump from the level $|i\alpha\rangle$ to the level $|f\alpha_1\rangle$ is

$$\begin{aligned} W(i\alpha \rightarrow f\alpha_1) = & \text{Tr}_D\{\langle f\alpha_1|\hat{U}_F(T - \tau)\hat{U}_M(\tau)(|i\alpha\rangle\langle i\alpha| \otimes |\Phi\rangle\langle\Phi|) \\ & \times \hat{U}_F^\dagger(T - \tau)\hat{U}_M^\dagger(\tau)|f\alpha_1\rangle\}. \end{aligned} \quad (4.85)$$

In the second-order approximation we obtain the expression for the jump probability $W(i\alpha \rightarrow f\alpha_1)$. The jump probability consists from three parts.

$$W(i\alpha \rightarrow f\alpha_1) = W_F(i\alpha \rightarrow f\alpha_1) + W_M(i\alpha \rightarrow f\alpha_1) + W_{\text{Int}}(i\alpha \rightarrow f\alpha_1), \quad (4.86)$$

where W_F is the probability of the jump during the free evolution, W_M is the probability of the jump during the measurement and W_{Int} is an interference term. The expressions for these probabilities are (see Refs. [47, 53] for the analogy of the derivation)

$$\begin{aligned} W_F(i\alpha \rightarrow f\alpha_1) = & \frac{1}{\hbar^2} \int_0^{T-\tau} dt_1 \int_0^{T-\tau} dt_2 V_{f\alpha_1, i\alpha}(t_1 + t_0 + \tau) V_{i\alpha, f\alpha_1}(t_2 + t_0 + \tau) \\ & \times \exp(i\omega_{f\alpha_1, i\alpha}(t_1 - t_2)), \end{aligned} \quad (4.87)$$

$$\begin{aligned} W_M(i\alpha \rightarrow f\alpha_1) = & \frac{1}{\hbar^2} \int_0^\tau dt_1 \int_0^\tau dt_2 V_{f\alpha_1, i\alpha}(t_1 + t_0) V_{i\alpha, f\alpha_1}(t_2 + t_0) \\ & \times \exp(i\omega_{f\alpha_1, i\alpha}(t_1 - t_2)) F(\lambda\omega_{fi}(t_1 - t_2)), \end{aligned} \quad (4.88)$$

$$\begin{aligned} W_{\text{Int}}(i\alpha \rightarrow f\alpha_1) = & \frac{2}{\hbar^2} \text{Re} \int_0^\tau dt_1 \int_\tau^T dt_2 V_{f\alpha_1, i\alpha}(t_1 + t_0) V_{i\alpha, f\alpha_1}(t_2 + t_0) \\ & \times \exp(i\omega_{f\alpha_1, i\alpha}(t_1 - t_2)) F(\lambda\omega_{if}(\tau - t_1)), \end{aligned} \quad (4.89)$$

where

$$\omega_{fi} = \frac{1}{\hbar}(E_f - E_i), \quad (4.90)$$

$$\omega_{f\alpha_1, i\alpha} = \omega_{fi} + \frac{1}{\hbar}(E_{\alpha_1} - E_\alpha), \quad (4.91)$$

$$F(x) = \langle\Phi|\exp(ix\hat{q})|\Phi\rangle. \quad (4.92)$$

The probability to remain for the system in the initial state $|i\alpha\rangle$ is

$$W(i\alpha) = 1 - \sum_{f,\alpha_1} W(i\alpha \rightarrow f\alpha_1). \quad (4.93)$$

After N measurements the probability for the system to survive in the initial state is equal to $W(i\alpha)^N \approx \exp(-RNT)$, where R is the measurement-modified decay rate

$$R = \sum_{f,\alpha_1} \frac{1}{T} W(i\alpha \rightarrow f\alpha_1) \quad (4.94)$$

4.4.2 Example: two-level system

As an example we will consider the evolution of the measured two-level system. The system is forced by the periodic of the frequency ω_L perturbation $V(t)$ which induces the jumps from one state to another. Such a system was used in the experiment by Itano *et al* [13]. The Hamiltonian of this system is

$$\hat{H} = \hat{H}_0 + \hat{V}(t) \quad (4.95)$$

where

$$\hat{H}_0 = \frac{\hbar\omega}{2} \hat{\sigma}_3, \quad (4.96)$$

$$\hat{V}(t) = (v\hat{\sigma}_+ + v^*\hat{\sigma}_-) \cos(\omega_L t). \quad (4.97)$$

Here $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices and $\sigma_{\pm} = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$. The Hamiltonian \hat{H}_0 has two eigenfunctions $|0\rangle$ and $|1\rangle$ with the eigenvalues $-\hbar\frac{\omega}{2}$ and $\hbar\frac{\omega}{2}$ respectively.

Using Eqs. (4.87), (4.88) and (4.89) for the jump from the state $|0\rangle$ to the state $|1\rangle$ we obtain

$$W_F(0 \rightarrow 1) = \frac{|v|^2 \sin^2\left(\frac{\Delta\omega}{2}(T - \tau)\right)}{\hbar^2 (\Delta\omega)^2}, \quad (4.98)$$

$$W_M(0 \rightarrow 1) = \frac{\tau |v|^2}{2 \hbar^2} \operatorname{Re} \int_0^\tau F(\lambda\omega t) \exp(i\Delta\omega t) \left(1 - \frac{t}{\tau}\right) dt, \quad (4.99)$$

$$W_{\text{Int}}(0 \rightarrow 1) = \frac{|v|^2}{2\hbar^2} \operatorname{Re} \int_0^\tau dt_1 \int_\tau^T dt_2 \exp(i\Delta\omega(t_1 - t_2)) F(\lambda\omega(t_1 - \tau)), \quad (4.100)$$

where $\Delta\omega = \omega - \omega_L$ is the detuning. Equation (4.99) has been obtained in Ref. [47].

When λ is large, the function F varies rapidly and we can approximate expressions (4.99) and (4.100) as

$$W_M(0 \rightarrow 1) = \frac{\tau |v|^2}{2\Lambda\omega \hbar^2} \quad (4.101)$$

$$W_{\text{Int}}(0 \rightarrow 1) = \frac{|v|^2}{\hbar^2} \frac{1}{2\Lambda\omega\Delta\omega} \sin(\Delta\omega(T - \tau)) \quad (4.102)$$

where $\Lambda = \lambda/C$, C is the width of the function F , defined by the equation (see Ref. [47])

$$C = \frac{1}{2} \int_{-\infty}^{\infty} F(x) dx \quad (4.103)$$

If $T \gg \tau$ and $\Delta\omega T \ll 1$ then we obtain

$$W(0 \rightarrow 1) = \frac{|v|^2 T^2}{\hbar^2} \frac{1}{4} + \frac{|v|^2 T}{\hbar^2} \frac{1}{2} \left(\frac{1}{\Lambda\omega} - \tau \right). \quad (4.104)$$

From Eq. (4.104) we see that the jump probability for the non-ideal measurement consists of two terms. The first term equals to the jump probability when the measurement is instantaneous, the second term represents the correction due to the finite duration of the measurement. In Ref. [47] it has been shown that the duration of the measurement can be estimated as

$$\tau \gtrsim \frac{1}{\Lambda\omega}. \quad (4.105)$$

From Eq. (4.104) we see that the correction term is small, since the duration of the measurement τ is almost compensated by the term $1/\Lambda\omega$.

4.5 General expression for the quantum Zeno and anti-Zeno effects

In this section we analyze the quantum Zeno and anti-Zeno effects without using any particular measurement model and making only few assumptions. We obtain a more general expression for the jump probability during the measurement. Expression, derived in Ref. [26] is a special case of our formula.

The measured system is described in section 4.2. The initial density matrix of the system is $\hat{\rho}_S(0)$. The initial density matrix of the detector is $\hat{\rho}_D(0)$. Before the measurement the measured system and the detector are uncorrelated, therefore, the full density matrix of the measured system and the detector is $\hat{\rho}(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_D(0)$. The duration of the measurement is τ .

When the interaction of the detector with the environment is taken into account, the evolution of the measured system and the detector cannot be described by a unitary operator. More general description of the evolution, allowing to include the interaction with the environment, can be given using the superoperators. Therefore, we will assume that the evolution of the measured system and the detector is given by the superoperator $\mathcal{S}(t)$. The explicit form of the superoperator $\mathcal{S}(t)$ can be obtained from a concrete model of the measurement.

Due to the finite duration of the measurement it is impossible to realize the infinitely frequent measurements. The highest frequency of the measurements is achieved when the measurements are performed one after another without the period of the measurement-free evolution between two successive measurements. Therefore, we model a continuous measurement by the subsequent measurements of the finite duration and finite accuracy. After N measurements the full density matrix of the measured system and the detector is

$$\hat{\rho}(N\tau) = \mathcal{S}(\tau)^N \hat{\rho}(0). \quad (4.106)$$

We assume that the density matrix of the detector $\hat{\rho}_D(0)$ is the same before each measurement. This means that the initial condition for the detector modified by the measurement is restored at the beginning of each measurement or each measurement is performed with new detector. For example, if the detector is an atom which is excited during the measurement, then after the interaction with the measured system is switched off, due to spontaneous emission the atom returns to the ground state and the result of the measurement is encoded in the emitted photon, thus the initial state of the detector is restored. In this case the full duration τ must be greater than the lifetime of the excited level.

4.5.1 Measurement of the unperturbed system

In this section we investigate the measurement of the unperturbed system, i.e., the case when $V(t) = 0$.

We assume that the measurement of the unperturbed system is a quantum non-demolition measurement [54–57]. The measurement of the unperturbed system does not change the state of the measured system when initially the system is in an eigenstate of the Hamiltonian \hat{H}_0 . After such an assumption, the most general form of the action of the superoperator $\mathcal{S}(\tau)$ can be written as

$$\mathcal{S}(\tau)[|n\alpha\rangle\langle m\alpha'| \otimes \hat{\rho}_D(0)] = |n\alpha\rangle\langle m\alpha'| e^{i\omega_{m\alpha',n\alpha}\tau} \otimes \mathcal{S}_{n\alpha,m\alpha'}(\tau) \hat{\rho}_D(0), \quad (4.107)$$

where

$$\omega_{m\alpha',n\alpha} = \frac{1}{\hbar}(E_m + E_{\alpha'} - E_n - E_{\alpha}) \quad (4.108)$$

and the superoperator $\mathcal{S}_{n\alpha,m\alpha'}(\tau)$ acts only on the density matrix of the detector. The full density matrix of the detector and the measured system after the measurement is

$$\hat{\rho}(\tau) = \mathcal{S}(\tau)\hat{\rho}(0) = \sum_{n\alpha,m\alpha'} |n\alpha\rangle(\rho_S)_{n\alpha,m\alpha'} e^{i\omega_{m\alpha',n\alpha}\tau} \langle m\alpha'| \otimes \mathcal{S}_{n\alpha,m\alpha'}(\tau) \hat{\rho}_D(0). \quad (4.109)$$

From Eq. (4.109) it follows that the non-diagonal matrix elements of the density matrix of the system after the measurement $(\rho_S)_{n\alpha,m\alpha'}(\tau)$ are multiplied by the quantity

$$F_{n\alpha,m\alpha'}(\tau) \equiv \text{Tr}\{\mathcal{S}_{n\alpha,m\alpha'}(\tau)\hat{\rho}_D(0)\}. \quad (4.110)$$

Since after the measurement the non-diagonal matrix elements of the density matrix of the measured system should become small (they must vanish in the case of an ideal measurement), $F_{n\alpha,m\alpha'}(\tau)$ must be also small when $n \neq m$.

4.5.2 Measurement of the perturbed system

The operator $\hat{V}(t)$ represents the perturbation of the unperturbed Hamiltonian $\hat{H}_0 + \hat{H}_1$. We will take into account the influence of the operator $\hat{V}(t)$ by the perturbation method, assuming that the strength of the interaction between the system and detector is large and the duration of the measurement τ is short. Similar method was used in Ref. [58].

We assume that the Markovian approximation is valid, i.e., the evolution of the measured system and the detector depends only on their state at the present time. Then the

superoperator \mathcal{S} , describing the evolution of the measured system and the detector, obeys the equation

$$\frac{\partial}{\partial t}\mathcal{S} = \mathcal{L}(t)\mathcal{S}, \quad (4.111)$$

where \mathcal{L} is the Liouvillian. There is a small perturbation of the measured system, given by the operator \hat{V} . We can write $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_V$, where \mathcal{L}_V is a small perturbation. We expand the superoperator \mathcal{S} into powers of V

$$\mathcal{S} = \mathcal{S}^{(0)} + \mathcal{S}^{(1)} + \mathcal{S}^{(2)} + \dots \quad (4.112)$$

Then from Eq. (4.111) it follows

$$\frac{\partial}{\partial t}\mathcal{S}^{(0)} = \mathcal{L}_0(t)\mathcal{S}^{(0)}, \quad (4.113)$$

$$\frac{\partial}{\partial t}\mathcal{S}^{(i)} = \mathcal{L}_0(t)\mathcal{S}^{(i)} + \mathcal{L}_V(t)\mathcal{S}^{(i-1)}. \quad (4.114)$$

We will denote as $\mathcal{S}^{(0)}(t, t_0)$ the solution of Eq. (4.113) with the initial condition $\mathcal{S}^{(0)}(t = t_0, t_0) = 1$. The formal solutions of Eqs. (4.113) and (4.114) are

$$\mathcal{S}^{(0)}(t, t_0) = T \exp \left(\int_{t_0}^t \mathcal{L}_0(t') dt' \right) \quad (4.115)$$

and

$$\mathcal{S}^{(i)}(t, 0) = \int_0^t dt_1 \mathcal{S}^{(0)}(t, t_1) \mathcal{L}_V(t_1) \mathcal{S}^{(i-1)}(t_1, 0). \quad (4.116)$$

Here T represents the time-ordering. In the second-order approximation we have

$$\begin{aligned} \mathcal{S}(t, 0) &= \mathcal{S}^{(0)}(t, 0) + \int_0^t dt_1 \mathcal{S}^{(0)}(t, t_1) \mathcal{L}_V(t_1) \mathcal{S}^{(0)}(t_1, 0) \\ &+ \int_0^t dt_1 \int_0^{t_1} dt_2 \mathcal{S}^{(0)}(t, t_1) \mathcal{L}_V(t_1) \mathcal{S}^{(0)}(t_1, t_2) \mathcal{L}_V(t_2) \mathcal{S}^{(0)}(t_2, 0). \end{aligned} \quad (4.117)$$

Using Eq. (4.112), the full density matrix of the measured system and the detector can be represented as

$$\hat{\rho}(t) = \hat{\rho}^{(0)}(t) + \hat{\rho}^{(1)}(t) + \hat{\rho}^{(2)}(t) + \dots, \quad (4.118)$$

where

$$\hat{\rho}^{(i)}(t) = \mathcal{S}^{(i)}(t, 0) \hat{\rho}(0). \quad (4.119)$$

Let the initial density matrix of the system and detector is

$$\hat{\rho}(0) = |i\alpha\rangle\langle i\alpha| \otimes \hat{\rho}_D(0). \quad (4.120)$$

The probability of the jump from the level $|i\alpha\rangle$ into the level $|f\alpha'\rangle$ during the measurement is

$$W(i\alpha \rightarrow f\alpha') = \text{Tr}\{|f\alpha'\rangle\langle f\alpha'| \hat{\rho}(\tau)\}. \quad (4.121)$$

Using the equation (4.107) we can write

$$\mathcal{S}^{(0)}(t, t_0) [|n\alpha\rangle\langle m\alpha'| \otimes \hat{\rho}_D(0)] = |n\alpha\rangle\langle m\alpha'| e^{i\omega_{m\alpha', n\alpha}(t-t_0)} \otimes \mathcal{S}_{n\alpha, m\alpha'}^{(0)}(t, t_0) \hat{\rho}_D(0). \quad (4.122)$$

4.5 General expression for the quantum Zeno and anti-Zeno effects

From Eq. (4.122) it follows that the superoperator $\mathcal{S}_{m\alpha, m\alpha}^{(0)}$ with the equal indices does not change the trace of the density matrix $\hat{\rho}_D$, since the trace of the full density matrix of the measured system and the detector must remain unchanged during the evolution.

When the system is perturbed by the operator $\hat{V}(t)$ then the superoperator \mathcal{L}_V is defined by the equation

$$\mathcal{L}_V(t)\hat{\rho} = \frac{1}{i\hbar}[\hat{V}(t), \hat{\rho}]. \quad (4.123)$$

The first-order term is $\hat{\rho}^{(1)}(t) = \mathcal{S}^{(1)}(t, 0)\hat{\rho}(0)$. Using Eqs. (4.116), (4.120), (4.122), and (4.123), this term can be written as

$$\begin{aligned} \hat{\rho}^{(1)}(t) = & \sum_{p\alpha_1} \frac{1}{i\hbar} \int_0^t dt_2 \left(|p\alpha_1\rangle V_{p\alpha_1, i\alpha}(t_2) e^{i\omega_{i\alpha, p\alpha_1}(t-t_2)} \langle i\alpha| \otimes \mathcal{S}_{p\alpha_1, i\alpha}^{(0)}(t, t_2) \right. \\ & \left. - |i\alpha\rangle V_{i\alpha, p\alpha_1}(t_2) e^{i\omega_{p\alpha_1, i\alpha}(t-t_2)} \langle p\alpha_1| \otimes \mathcal{S}_{i\alpha, p\alpha_1}^{(0)}(t, t_2) \right) \mathcal{S}_{i\alpha, i\alpha}^{(0)}(t_2, 0) \hat{\rho}_D(0). \end{aligned} \quad (4.124)$$

When $i \neq f$ then the first-order term does not contribute to the jump probability, since from Eqs. (4.121) and (4.124) it follows that the expression for this contribution contains the scalar product $\langle f\alpha' | i\alpha \rangle = 0$.

For the second-order term $\hat{\rho}^{(2)}(t) = \mathcal{S}^{(2)}(t, 0)\hat{\rho}(0)$, using Eqs. (4.116) and (4.122), we obtain the equality

$$\text{Tr}\{|f\alpha'\rangle \langle f\alpha'| \hat{\rho}^{(2)}(t)\} = \frac{1}{i\hbar} \int_0^t dt_1 \text{Tr} \left\{ \langle f\alpha' | \hat{V}(t_1) \hat{\rho}^{(1)}(t_1) | f\alpha' \rangle - \langle f\alpha' | \hat{\rho}^{(1)}(t_1) \hat{V}(t_1) | f\alpha' \rangle \right\}. \quad (4.125)$$

In Eq. (4.125) the superoperator $\mathcal{S}_{f\alpha', f\alpha'}^{(0)}$ is omitted, since it does not change the trace. Then from Eqs. (4.124) and (4.125) we obtain the jump probability

$$\begin{aligned} W(i\alpha \rightarrow f\alpha') = & \frac{1}{\hbar^2} \int_0^\tau dt_1 \int_0^{t_1} dt_2 \text{Tr} \left\{ \left(V_{f\alpha', i\alpha}(t_1) V_{i\alpha, f\alpha'}(t_2) \mathcal{S}_{i\alpha, f\alpha'}^{(0)}(t_1, t_2) e^{i\omega_{f\alpha', i\alpha}(t_1-t_2)} \right. \right. \\ & \left. \left. + V_{f\alpha', i\alpha}(t_2) V_{i\alpha, f\alpha'}(t_1) \mathcal{S}_{f\alpha', i\alpha}^{(0)}(t_1, t_2) e^{i\omega_{i\alpha, f\alpha'}(t_1-t_2)} \right) \right. \\ & \left. \times \mathcal{S}_{i\alpha, i\alpha}^{(0)}(t_2, 0) \hat{\rho}_D(0) \right\}. \end{aligned} \quad (4.126)$$

Equation (4.126) allows us to calculate the jump probability during the measurement when the evolution of the measured unperturbed system is known. The explicit form of the superoperator $\mathcal{S}_{n\alpha, m\alpha'}^{(0)}$ can be obtained from a concrete model of the measurement. The main assumptions, used in the derivation of Eq. (4.126), are Eqs. (4.107) and (4.111), i.e., the assumptions that the quantum measurement of the unperturbed system is non-demolition measurement and that the Markovian approximation is valid. Thus, Eq. (4.126) is quite general.

The probability that the measured system remains in the initial state $|i\alpha\rangle$ is

$$W(i\alpha) = 1 - \sum_{f, \alpha'} W(i\alpha \rightarrow f\alpha'). \quad (4.127)$$

After N measurements the probability that the measured system remains in the initial state equals to

$$W(i\alpha)^N \approx \exp(-RN\tau), \quad (4.128)$$

where R is the jump rate

$$R = \sum_{f,\alpha'} \frac{1}{\tau} W(i\alpha \rightarrow f\alpha'). \quad (4.129)$$

4.5.3 Free evolution and measurements

In practice, it is impossible to perform the measurements one after another without the period of the measurement-free evolution between two successive measurements. Such intervals of the measurement-free evolution were also present in the experiments demonstrating the quantum Zeno effect [13, 18, 21]. Therefore, it is important to consider such measurements. This problem for the definite model was investigated in Ref. [59].

We have the repeated measurements separated by the free evolution of the measured system. For the purpose of the description of such measurements we can use Eq. (4.126), obtained in Sec. 4.5.2. The duration of the free evolution is τ_F , the duration of the free evolution and the measurement together is τ . The superoperator of the free evolution without the perturbation \hat{V} is $\mathcal{S}_F^{(0)}(t)$, the superoperator of the measurement is $\mathcal{S}_M^{(0)}(t, t_0)$. We will assume that during the measurement the superoperator \mathcal{L}_0 does not depend on time t . Then the superoperator $\mathcal{S}_M^{(0)}(t, t_0)$ depends only on the time difference $t - t_0$. Therefore, we will write $\mathcal{S}_M^{(0)}(t - t_0)$ instead of $\mathcal{S}_M^{(0)}(t, t_0)$. When the free evolution comes first and then the measurement is performed, the full superoperator equals to

$$\mathcal{S}_{n\alpha, m\alpha'}^{(0)}(t, t_1) = \begin{cases} \mathcal{S}_{M n\alpha, m\alpha'}^{(0)}(t - t_1), & \tau > t_1 > \tau_F \text{ and } \tau > t > t_1, \\ \mathcal{S}_F^{(0)}(t - t_1), & \tau_F > t_1 > 0 \text{ and } \tau_F > t > t_1, \\ \mathcal{S}_{M n\alpha, m\alpha'}^{(0)}(t - \tau_F) \mathcal{S}_F^{(0)}(\tau_F - t_1), & \tau_F > t_1 > 0 \text{ and } \tau > t > \tau_F. \end{cases} \quad (4.130)$$

Equation (4.130) can be written as

$$\begin{aligned} \mathcal{S}_{n\alpha, m\alpha'}^{(0)}(t, t_1) &= \mathcal{S}_{M n\alpha, m\alpha'}^{(0)}(t - t_1) \Theta(t_1 - \tau_F) + \mathcal{S}_F^{(0)}(t - t_1) \Theta(\tau_F - t) \\ &\quad + \mathcal{S}_{M n\alpha, m\alpha'}^{(0)}(t - \tau_F) \mathcal{S}_F^{(0)}(\tau_F - t_1) \Theta(t - \tau_F) \Theta(\tau_F - t_1), \end{aligned} \quad (4.131)$$

where Θ is Heaviside unit step function. From Eqs. (4.126) and (4.131) it follows that the jump probability consists of three terms

$$W(i\alpha \rightarrow f\alpha') = W_M(i\alpha \rightarrow f\alpha') + W_F(i\alpha \rightarrow f\alpha') + W_I(i\alpha \rightarrow f\alpha'), \quad (4.132)$$

where the jump probability during the free evolution is

$$W_F(i\alpha \rightarrow f\alpha') = \frac{1}{\hbar^2} \int_0^{\tau_F} dt_1 \int_0^{\tau_F} dt_2 V_{f\alpha', i\alpha}(t_1) V_{i\alpha, f\alpha'}(t_2) e^{i\omega_{f\alpha', i\alpha}(t_1 - t_2)}, \quad (4.133)$$

the jump probability during the measurement

$$\begin{aligned} W_M(i\alpha \rightarrow f\alpha') &= \frac{1}{\hbar^2} \int_{\tau_F}^{\tau} dt_1 \int_{\tau_F}^{t_1} dt_2 \text{Tr} \left\{ \left(V_{f\alpha', i\alpha}(t_1) V_{i\alpha, f\alpha'}(t_2) \mathcal{S}_{M i\alpha, f\alpha'}^{(0)}(t_1 - t_2) e^{i\omega_{f\alpha', i\alpha}(t_1 - t_2)} \right. \right. \\ &\quad \left. \left. + V_{f\alpha', i\alpha}(t_2) V_{i\alpha, f\alpha'}(t_1) \mathcal{S}_{M f\alpha', i\alpha}^{(0)}(t_1 - t_2) e^{i\omega_{i\alpha, f\alpha'}(t_1 - t_2)} \right) \right. \\ &\quad \left. \times \mathcal{S}_{M i\alpha, i\alpha}^{(0)}(t_2 - \tau_F) \mathcal{S}_F^{(0)}(\tau_F) \hat{\rho}_D(0) \right\}, \end{aligned} \quad (4.134)$$

and the interference term is

$$\begin{aligned}
 W_I(i\alpha \rightarrow f\alpha') &= \frac{1}{\hbar^2} \int_{\tau_F}^{\tau} dt_1 \int_0^{\tau_F} dt_2 \text{Tr} \left\{ \left(V_{f\alpha',i\alpha}(t_1) V_{i\alpha,f\alpha'}(t_2) \mathcal{S}_{M i\alpha,f\alpha'}^{(0)}(t_1 - \tau_F) e^{i\omega_{f\alpha',i\alpha}(t_1-t_2)} \right. \right. \\
 &\quad \left. \left. + V_{f\alpha',i\alpha}(t_2) V_{i\alpha,f\alpha'}(t_1) \mathcal{S}_{M f\alpha',i\alpha}^{(0)}(t_1 - \tau_F) e^{i\omega_{i\alpha,f\alpha'}(t_1-t_2)} \right) \mathcal{S}_F^{(0)}(\tau_F) \hat{\rho}_D(0) \right\}. \tag{4.135}
 \end{aligned}$$

If we assume that the free evolution does not change the density matrix of the detector and the perturbation \hat{V} does not depend on time, we have the jump probability during the measurement-free evolution

$$W_F(i\alpha \rightarrow f\alpha') = |V_{i\alpha,f\alpha'}|^2 \frac{4 \sin^2 \left(\frac{1}{2} \omega_{f\alpha',i\alpha} \tau_F \right)}{\hbar^2 \omega_{f\alpha',i\alpha}^2}, \tag{4.136}$$

the jump probability during the measurement

$$\begin{aligned}
 W_M(i\alpha \rightarrow f\alpha') &= \frac{1}{\hbar^2} |V_{i\alpha,f\alpha'}|^2 \int_{\tau_F}^{\tau} dt_1 \int_{\tau_F}^{t_1} dt_2 \text{Tr} \left\{ \left(\mathcal{S}_{M i\alpha,f\alpha'}^{(0)}(t_1 - t_2) e^{i\omega_{f\alpha',i\alpha}(t_1-t_2)} \right. \right. \\
 &\quad \left. \left. + \mathcal{S}_{M f\alpha',i\alpha}^{(0)}(t_1 - t_2) e^{i\omega_{i\alpha,f\alpha'}(t_1-t_2)} \right) \right. \\
 &\quad \left. \times \mathcal{S}_{M i\alpha,i\alpha}^{(0)}(t_2 - \tau_F) \hat{\rho}_D(0) \right\}, \tag{4.137}
 \end{aligned}$$

and the interference term

$$\begin{aligned}
 W_I(i\alpha \rightarrow f\alpha') &= |V_{i\alpha,f\alpha'}|^2 \frac{2 \sin \left(\frac{1}{2} \omega_{f\alpha',i\alpha} \tau_F \right)}{\hbar^2 \omega_{f\alpha',i\alpha}} \\
 &\quad \times \int_{\tau_F}^{\tau} dt_1 \text{Tr} \left\{ \left(\mathcal{S}_{M i\alpha,f\alpha'}^{(0)}(t_1 - \tau_F) e^{i\omega_{f\alpha',i\alpha}(t_1 - \frac{1}{2} \tau_F)} \right. \right. \\
 &\quad \left. \left. + \mathcal{S}_{M f\alpha',i\alpha}^{(0)}(t_1 - \tau_F) e^{i\omega_{i\alpha,f\alpha'}(t_1 - \frac{1}{2} \tau_F)} \right) \hat{\rho}_D(0) \right\}. \tag{4.138}
 \end{aligned}$$

4.5.4 Simplification of the expression for the jump probability

The expression for the jump probability during the measurement can be simplified if the operator \hat{V} does not depend on time t . Then Eq. (4.126) can be written as

$$W(i\alpha \rightarrow f\alpha') = \frac{2}{\hbar^2} |V_{i\alpha,f\alpha'}|^2 \text{Re} \int_0^{\tau} dt_1 \int_0^{t_1} dt_2 e^{i\omega_{f\alpha',i\alpha}(t_1-t_2)} \text{Tr} \{ \mathcal{S}_{i\alpha,f\alpha'}^{(0)}(t_1, t_2) \mathcal{S}_{i\alpha,i\alpha}^{(0)}(t_2, 0) \hat{\rho}_D(0) \}. \tag{4.139}$$

Introducing the function

$$G_{f\alpha',i\alpha}(\omega) = |V_{i\alpha,f\alpha'}|^2 \delta \left(\frac{1}{\hbar} (E_{\alpha'} - E_{\alpha}) - \omega \right) \tag{4.140}$$

we can rewrite Eq. (4.139) in the form

$$W(i\alpha \rightarrow f\alpha') = \frac{2\pi\tau}{\hbar^2} \int_{-\infty}^{\infty} G_{f\alpha',i\alpha}(\omega) P_{i\alpha,f\alpha'}(\omega) d\omega, \tag{4.141}$$

where

$$P(\omega)_{i\alpha, f\alpha'} = \frac{1}{\pi\tau} \operatorname{Re} \int_0^\tau dt_1 \int_0^{t_1} dt_2 e^{i(\omega - \omega_{if})(t_1 - t_2)} \operatorname{Tr} \{ \mathcal{S}_{i\alpha, f\alpha'}^{(0)}(t_1, t_2) \mathcal{S}_{i\alpha, i\alpha}^{(0)}(t_2, 0) \hat{\rho}_D(0) \}. \quad (4.142)$$

Equation (4.141) is similar to that obtained by Kofman and Kurizki in Ref. [26].

Further simplification can be achieved when the superoperator \mathcal{L}_0 does not depend on time t and the order of the superoperators in the expression $\operatorname{Tr} \{ \mathcal{S}_{i\alpha, f\alpha'}^{(0)}(t_1, t_2) \mathcal{S}_{i\alpha, i\alpha}^{(0)}(t_2) \hat{\rho}_D(0) \}$ can be changed. Under such assumptions we have

$$\operatorname{Tr} \{ \mathcal{S}_{i\alpha, f\alpha'}^{(0)}(t_1, t_2) \mathcal{S}_{i\alpha, i\alpha}^{(0)}(t_2) \hat{\rho}_D(0) \} = \operatorname{Tr} \{ \mathcal{S}_{i\alpha, i\alpha}^{(0)}(t_2) \mathcal{S}_{i\alpha, f\alpha'}^{(0)}(t_1, t_2) \hat{\rho}_D(0) \} = F_{i\alpha, f\alpha'}(t_1 - t_2), \quad (4.143)$$

where $F_{i\alpha, f\alpha'}(t)$ is defined by Eq. (4.110). After changing the variables into $u = t_1 - t_2$ and $v = t_1 + t_2$ from Eq. (4.142) we obtain

$$P(\omega)_{i\alpha, f\alpha'} = \frac{1}{\pi} \operatorname{Re} \int_0^\tau \left(1 - \frac{u}{\tau}\right) F_{i\alpha, f\alpha'}(u) \exp(i(\omega - \omega_{if})u) du. \quad (4.144)$$

Decaying system

We consider a decaying system with the Hamiltonian \hat{H}_0 which due to the interaction with the field decays from the level $|i\rangle$ into the level $|f\rangle$. The field initially is in the vacuum state $|\alpha = 0\rangle$. Only the energy levels of the decaying system are measured and the detector does not interact with the field. Then $\mathcal{S}_{i\alpha, f\alpha'}^{(0)}$ and $P_{i\alpha, f\alpha'}(\omega)$ do not depend on α and α' . Using Eqs. (4.129) and (4.141) we obtain the decay rate of the measured system

$$R = \sum_\alpha \frac{1}{\tau} W(i0 \rightarrow f\alpha) = \frac{2\pi}{\hbar^2} \int_{-\infty}^{\infty} G_{f,i}(\omega) P_{i,f}(\omega) d\omega, \quad (4.145)$$

where

$$G_{f,i}(\omega) = \sum_\alpha G_{f\alpha, i0}(\omega). \quad (4.146)$$

The function $P_{i,f}(\omega)$ is related to the measurement-induced broadening of the spectral line [26, 47, 60]. For example, when the instantaneous ideal measurements are performed at time intervals τ , we can take $F_{i\alpha, f\alpha'}(t) = \Theta(\tau - t)$, where $\Theta(t)$ is the unit step function. Then from Eq. (4.144) we get

$$P_{i,f}(\omega) = \frac{2 \sin^2\left(\frac{1}{2}\tau(\omega - \omega_{if})\right)}{\pi\tau(\omega - \omega_{if})^2}.$$

We have that the width of the function $P_{i,f}(\omega)$ increases when the duration of the measurement τ decreases.

Equation (4.141) represents a universal result: the decay rate of the frequently measured decaying system is determined by the overlap of the reservoir coupling spectrum $G_{f,i}(\omega)$ and the measurement-modified level width $P_{i,f}(\omega)$.

Depending on the reservoir spectrum $G_{f,i}(\omega)$ and the frequency of the measurements $1/\tau$ the inhibition or acceleration of the decay can be obtained. If the frequency of the

measurements is small and, consequently, the measurement-induced broadening of the spectral line is much smaller than the width of the reservoir coupling spectrum, the decay rate equals the decay rate of the unmeasured system, given by the Fermi's Golden Rule. In the intermediate region, when the width of the spectral line is rather small compared with the distance between ω_{if} and the nearest maximum in the reservoir spectrum, the decay rate grows with the increase of the frequency of the measurements. This results in the anti-Zeno effect.

If the width of the spectral line is much greater compared both with the width of the reservoir spectrum and the distance between ω_{if} and the centrum of the reservoir spectrum, the decay rate decreases when the frequency of measurements increases. This results in the quantum Zeno effect.

4.6 Influence of the detector's temperature on the quantum Zeno effect

In this section we extend the model, used in Ref. [47], including the interaction of the detector with the environment. Then it becomes possible to study the influence of other parameters of the detector on the evolution of the measured system, too. Using our model we analyze the influence of the detector's temperature on the measured system.

To describe the decoherence and dissipation we use the the Lindblad-type master equation. Semigroup theories pioneered by Lindblad [61] demonstrated that density-matrix positivity, translational invariance and approach to thermal equilibrium cannot be satisfied simultaneously. Under the assumption of Markovian dynamics and initial decoupling of system and bath, the semigroup approach adds dissipative dynamics to the quantum master equations by means of the Lindblad dissipation operators.

Recently the semigroup formalism has attracted much attention. Quantum computing is one of the fields in which quantum dissipation finds the most recent applications. In physical chemistry semigroup theories have been utilized to model dynamics of ultrafast predissociation in a condensed-phase or cluster environment [62], and electronic quenching due to the coupling of the adsorbate negative ion in resonance to the metal electrons in the desorption of neutral molecules on metal surfaces [63]. In nuclear physics, the semigroup formalism is applied to model giant resonances in the nuclear spectra above the neutron emission threshold [64].

4.6.1 Model of the measurement

The measured system is described in section 4.2. As the detector we use a harmonic oscillator with the Hamiltonian

$$\hat{H}_D = \hbar\Omega \left(\hat{b}^\dagger \hat{b} + \frac{1}{2} \right), \quad (4.147)$$

where \hat{b} and \hat{b}^\dagger are the creation and anihilation operators, respectively. We choose the interaction operator \hat{H}_I in the form

$$\hat{H}_I = \lambda \hat{q} \hat{H}_0, \quad (4.148)$$

where $\hat{q} = \hat{b}^\dagger + \hat{b}$ is the coordinate of the detector and the parameter λ describes the strength of the interaction. This system — detector interaction is similar to that considered by von Neumann [5] and in Refs. [43–47, 53]. In order to obtain a sensible measurement, the parameter λ must be large.

The measurement begins at time moment t_0 . At the beginning of the interaction with the detector, the detector's density matrix is $\hat{\rho}_D(t_0)$. The detector initially is in the thermal equilibrium with the temperature T . Therefore,

$$\hat{\rho}_D(t_0) = \hat{\rho}_T = \exp\left(-\frac{\hbar\Omega\hat{n}}{k_B T}\right) \left(1 - \exp\left(-\frac{\hbar\Omega}{k_B T}\right)\right), \quad (4.149)$$

where $\hat{n} = \hat{b}^\dagger\hat{b}$. The average excitation of the detector in thermal equilibrium with the temperature T is

$$\bar{n}(T) = \left(\exp\left(\frac{\hbar\Omega}{k_B T}\right) - 1\right)^{-1}. \quad (4.150)$$

The full density matrix of the system and detector is $\hat{\rho}(t_0) = \hat{\rho}_S(t_0) \otimes \hat{\rho}_D(t_0)$ where $\hat{\rho}_S(t_0)$ is the density matrix of the system.

The detector is interacting with the environment. The master equation for the density matrix of the system and the detector in the Lindblad form is (Ref. [61])

$$\frac{\partial\hat{\rho}(t)}{\partial t} = \frac{1}{i\hbar}[\hat{H}, \hat{\rho}(t)] + L_D[\hat{\rho}(t)], \quad (4.151)$$

where

$$L_D[\hat{\rho}(t)] = \sum_{\mu} ([\hat{V}_{\mu}\hat{\rho}(t), \hat{V}_{\mu}^\dagger] + [\hat{V}_{\mu}, \hat{\rho}(t)\hat{V}_{\mu}^\dagger]), \quad (4.152)$$

and \hat{V}_{μ} are the Lindblad dissipation operators. We use the equation of a dissipative phase damped oscillator discussed in quantum optic [4]. The Lindblad dissipation operators are chosen as follows,

$$\hat{V}_1 = \sqrt{\frac{\gamma}{2}}\hat{b}^\dagger\hat{b}, \quad \hat{V}_2 = \sqrt{\frac{\gamma_{\uparrow}}{2}}\hat{b}^\dagger, \quad \hat{V}_3 = \sqrt{\frac{\gamma_{\downarrow}}{2}}\hat{b}. \quad (4.153)$$

Then the equation (4.151) for the density matrix becomes

$$\begin{aligned} \frac{\partial\hat{\rho}(t)}{\partial t} &= \frac{1}{i\hbar}[\hat{H}, \hat{\rho}(t)] + \frac{\gamma}{2}(2\hat{n}\hat{\rho}(t)\hat{n} - \hat{n}^2\hat{\rho}(t) - \hat{\rho}(t)\hat{n}^2) \\ &+ \frac{\gamma_{\uparrow}}{2}(2\hat{b}^\dagger\hat{\rho}(t)\hat{b} - (\hat{n} + 1)\hat{\rho}(t) - \hat{\rho}(t)(\hat{n} + 1)) \\ &+ \frac{\gamma_{\downarrow}}{2}(2\hat{b}\hat{\rho}(t)\hat{b}^\dagger - \hat{n}\hat{\rho}(t) - \hat{\rho}(t)\hat{n}). \end{aligned} \quad (4.154)$$

The approach to the thermal equilibrium is obtained when the parameters γ_{\uparrow} and γ_{\downarrow} satisfy the condition [65]

$$\gamma_{\uparrow} = \gamma_{\downarrow} \exp\left(-\frac{\hbar\Omega}{k_B T}\right). \quad (4.155)$$

4.6.2 Solution of the master equation

For the solution of the equation (4.154) we adopt the technique used in Ref. [66]. We introduce the quantum characteristic function [67]

$$\chi(\xi, \xi^*) = \text{Tr}\{\hat{\rho}e^{\xi\hat{b}^\dagger}e^{-\xi^*\hat{b}}\}. \quad (4.156)$$

The quantum characteristic function of the detector at the thermal equilibrium is

$$\chi_T(\xi, \xi^*) = \exp(-\xi\xi^*\bar{n}(T)). \quad (4.157)$$

We multiply the equation (4.151) by $\exp(-\xi^*\hat{b})$ from the left and by $\exp(\xi\hat{b}^\dagger)$ from the right and take the trace. When the interaction between the measured system and the detector is absent (i.e., $\lambda = 0$), we obtain the equation

$$\begin{aligned} \frac{\partial}{\partial t}\chi(\xi, \xi^*; t) &= i\Omega \left(\xi \frac{\partial}{\partial \xi} \chi - \xi^* \frac{\partial}{\partial \xi^*} \chi \right) \\ &+ \frac{\gamma}{2} \left(2\xi^*\xi \frac{\partial^2}{\partial \xi \partial \xi^*} \chi - \xi^{*2} \frac{\partial^2}{\partial \xi^{*2}} \chi - \xi^2 \frac{\partial^2}{\partial \xi^2} \chi - \xi^* \frac{\partial}{\partial \xi^*} \chi - \xi \frac{\partial}{\partial \xi} \chi \right) \\ &+ \frac{\gamma_\uparrow}{2} \left(\xi \frac{\partial}{\partial \xi} \chi + \xi^* \frac{\partial}{\partial \xi^*} \chi - 2\xi\xi^* \chi \right) \\ &- \frac{\gamma_\downarrow}{2} \left(\xi \frac{\partial}{\partial \xi} \chi + \xi^* \frac{\partial}{\partial \xi^*} \chi \right). \end{aligned} \quad (4.158)$$

We will search the solution of Eq. (4.158) in the form

$$\chi(\xi, \xi^*) = \exp \left(\sum_{j,k} C_{j,k}(t) \xi^j (-\xi^*)^k \right), \quad (4.159)$$

where $C_{j,k}$ are the coefficients to be determined. Substituting Eq. (4.159) into Eq. (4.158) we obtain the set of equations for the coefficients $C_{j,k}$

$$\begin{aligned} \frac{\partial C_{j,k}(t)}{\partial t} &= i\Omega(j-k)C_{j,k}(t) - \frac{\gamma}{2}(j-k)^2 C_{j,k}(t) \\ &+ \frac{1}{2}(\gamma_\uparrow - \gamma_\downarrow)(j+k)C_{j,k}(t) + \gamma_\uparrow \delta_{j,1} \delta_{k,1}. \end{aligned} \quad (4.160)$$

The solution of Eq. (4.160) is

$$C_{1,1}(t) = C_{1,1}(0)e^{-(\gamma_\downarrow - \gamma_\uparrow)t} + \bar{n}(T) (1 - e^{-(\gamma_\downarrow - \gamma_\uparrow)t}), \quad (4.161)$$

$$C_{j,k}(t) = C_{j,k}(0)e^{i\Omega(j-k)t} e^{-\frac{\gamma}{2}(j-k)^2 t - \frac{1}{2}(\gamma_\downarrow - \gamma_\uparrow)(j+k)t}, \quad j \neq 1, k \neq 1 \quad (4.162)$$

From the solution we see that the function χ approaches the function at the equilibrium χ_T as the time t grows. The detector's density matrix $\hat{\rho}_D$, correspondingly, tends to the $\hat{\rho}_T$.

4.6.3 Measurement of the unperturbed system

At first, we will consider the case when the perturbation is absent, i.e., $\hat{V}(t) = 0$. Since the Hamiltonian of the measured system does not depend on t we will omit the parameter t_0 in this section.

We introduce the density matrix $\hat{\rho}_{m,n} = \sum_{\alpha} \langle m\alpha | \hat{\rho} | n\alpha \rangle$ and the characteristic function

$$\chi_{m,n}(\xi, \xi^*; t) = \text{Tr}\{\hat{\rho}_{m,n}(t)e^{\xi\hat{b}^\dagger}e^{-\xi^*\hat{b}}\}. \quad (4.163)$$

From Eq. (4.151) we obtain the equation for the density matrix $\hat{\rho}_{m,n}$

$$\frac{\partial}{\partial t}\hat{\rho}_{m,n} = -i\omega_{mn}\hat{\rho}_{m,n} - i\lambda(\omega_m\hat{q}\hat{\rho}_{m,n} - \hat{\rho}_{m,n}\hat{q}\omega_n) + L_D[\hat{\rho}_{m,n}], \quad (4.164)$$

where

$$\omega_n = \frac{E_n}{\hbar}, \quad (4.165)$$

$$\omega_{mn} = \omega_m - \omega_n. \quad (4.166)$$

Equation (4.164) may be solved similarly as in Sec. 4.6.2. When the detector is initially at equilibrium then $\chi_{mn}(\xi, \xi^*; 0) = \chi_{mn}(0) \exp(-\xi\xi^*\bar{n}(T))$. As in Sec. 4.6.2 we take the characteristic function of the form (4.159) and obtain the equations for the coefficients $C_{j,k}$

$$\frac{\partial C_{0,0}}{\partial t} = -i\omega_{mn}(1 + \lambda(C_{1,0} + C_{0,1})), \quad (4.167)$$

$$\frac{\partial C_{1,0}}{\partial t} = (i\Omega - \gamma_{\text{eff}})C_{1,0} + i\lambda(\omega_n - \omega_{mn}C_{1,1}), \quad (4.168)$$

$$\frac{\partial C_{0,1}}{\partial t} = -(i\Omega + \gamma_{\text{eff}})C_{0,1} - i\lambda(\omega_m + \omega_{mn}C_{1,1}), \quad (4.169)$$

$$\frac{\partial C_{1,1}}{\partial t} = (\gamma_{\uparrow} - \gamma_{\downarrow})C_{1,1} + \gamma_{\uparrow} \quad (4.170)$$

with the initial conditions $C_{0,0}(0) = 0$, $C_{1,0}(0) = 0$, $C_{0,1}(0) = 0$, $C_{1,1}(0) = \bar{n}(T)$. Here

$$\gamma_{\text{eff}} = \frac{1}{2}(\gamma + \gamma_{\downarrow} - \gamma_{\uparrow}). \quad (4.171)$$

The solutions of Eqs. (4.167)–(4.170) are

$$C_{1,1}(t) = \bar{n}(T), \quad (4.172)$$

$$C_{1,0}(t) = i\lambda \frac{\omega_n - \omega_{mn}\bar{n}(T)}{\gamma_{\text{eff}} - i\Omega} (1 - e^{(i\Omega - \gamma_{\text{eff}})t}), \quad (4.173)$$

$$C_{0,1}(t) = -i\lambda \frac{\omega_m + \omega_{mn}\bar{n}(T)}{\gamma_{\text{eff}} + i\Omega} (1 - e^{-(i\Omega + \gamma_{\text{eff}})t}), \quad (4.174)$$

$$C_{0,0}(t) = -i\omega_{mn}t + \lambda^2\omega_{mn} \frac{\omega_n - \omega_{mn}\bar{n}(T)}{\gamma_{\text{eff}} - i\Omega} \left(t + \frac{1}{\gamma_{\text{eff}} - i\Omega} (e^{(i\Omega - \gamma_{\text{eff}})t} - 1) \right) - \lambda^2\omega_{mn} \frac{\omega_m + \omega_{mn}\bar{n}(T)}{\gamma_{\text{eff}} + i\Omega} \left(t + \frac{1}{\gamma_{\text{eff}} + i\Omega} (e^{-(i\Omega + \gamma_{\text{eff}})t} - 1) \right). \quad (4.175)$$

Using Eqs. (4.159) and (4.175) we find that the non-diagonal elements of the density matrix of the measured system become small as the time t grows. This represents the decoherence induced by the measurement. The diagonal elements of the density matrix do not change.

4.6.4 Measurement of the perturbed system

The operator $\hat{V}(t)$ represents the perturbation of the unperturbed Hamiltonian $\hat{H}_0 + \hat{H}_1$. We assume that diagonal matrix elements of the perturbation operator V are zeros. We will take into account the influence of the operator \hat{V} by the perturbation method, assuming that the strength of the interaction λ between the system and detector is large. This method was developed in Sec. 4.5.2.

Using Eq. (4.126) we obtain the probability of the jump during the measurement

$$W(i\alpha \rightarrow f\alpha_1, t) = \frac{1}{\hbar^2} |V_{i\alpha, f\alpha_1}|^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \text{Tr} \left\{ \left(S_{i\alpha, f\alpha_1}^{(0)}(t_1 - t_2) + S_{f\alpha_1, i\alpha}^{(0)}(t_1 - t_2) \right) S_{i\alpha, i\alpha}^{(0)}(t_2) \hat{\rho}_D \right\}. \quad (4.176)$$

Defining a new characteristic function similarly as in Eq. (4.163)

$$\chi_{i\alpha, f\alpha_1}(\xi, \xi^*; t_1, t_2) = \text{Tr} \{ e^{\xi \hat{b}^\dagger} e^{-\xi^* \hat{b}} S_{i\alpha, f\alpha_1}^{(0)}(t_1 - t_2) S_{i\alpha, i\alpha}^{(0)}(t_2) \hat{\rho}_D \} \quad (4.177)$$

the jump probability (4.176) can be expressed as

$$W(i\alpha \rightarrow f\alpha_1, t) = \frac{1}{\hbar^2} |V_{i\alpha, f\alpha_1}|^2 \int_0^t dt_1 \int_0^{t_1} dt_2 (\chi_{i\alpha, f\alpha_1}(0, 0; t_1, t_2) + \chi_{f\alpha_1, i\alpha}(0, 0; t_1, t_2)). \quad (4.178)$$

The detector initially (at $t = 0$) is in the thermal equilibrium with the temperature T , $\hat{\rho}_D = \hat{\rho}_T$ (Eq. (4.149)). The initial characteristic function is $\chi_{i\alpha, f\alpha_1}(\xi, \xi^*; 0, 0) = \exp(-\xi \xi^* \bar{n}(T))$. Using the results of the Sec. 4.6.3 (Eqs. (4.159) and (4.172)–(4.175)), we obtain the characteristic function of the density matrix at time t_2

$$\begin{aligned} \chi_{i\alpha, f\alpha_1}(\xi, \xi^*; t_2, t_2) &= \text{Tr} \{ e^{\xi \hat{b}^\dagger} e^{-\xi^* \hat{b}} S_{i\alpha, i\alpha}^{(0)}(t_2) \hat{\rho}_D \} \\ &= \exp \left(\frac{i\xi \lambda \omega_i}{\gamma_{\text{eff}} - i\Omega} (1 - e^{i(\Omega - \gamma_{\text{eff}})t_2}) \right) \\ &\times \exp \left(\frac{i\xi^* \lambda \omega_i}{\gamma_{\text{eff}} + i\Omega} (1 - e^{-i(\Omega + \gamma_{\text{eff}})t_2}) - \xi \xi^* \bar{n}(T) \right). \end{aligned} \quad (4.179)$$

Taking the function $\chi_{i\alpha, f\alpha_1}(\xi, \xi^*; t_2, t_2)$ from Eq. (4.179) as the initial characteristic function and proceeding further as in Sec. 4.6.3, we have the value of the characteristic function,

defined by Eq. (4.177) with the parameters $\xi = \xi^* = 0$

$$\begin{aligned} \chi_{i\alpha, f\alpha_1}(0, 0; t_1, t_2) = & \exp \left(-i\omega_{i\alpha, f\alpha_1}(t_1 - t_2) \right. \\ & + \lambda^2 \omega_{if} \frac{\omega_f - \omega_{if} \bar{n}(T)}{\gamma_{\text{eff}} - i\Omega} \left(t_1 - t_2 + \frac{1}{\gamma_{\text{eff}} - i\Omega} (e^{(i\Omega - \gamma_{\text{eff}})(t_1 - t_2)} - 1) \right) \\ & - \lambda^2 \omega_{if} \frac{\omega_i + \omega_{if} \bar{n}(T)}{\gamma_{\text{eff}} + i\Omega} \left(t_1 - t_2 + \frac{1}{\gamma_{\text{eff}} + i\Omega} (e^{-(i\Omega + \gamma_{\text{eff}})(t_1 - t_2)} - 1) \right) \\ & + \frac{\lambda^2 \omega_{if} \omega_i}{(\gamma_{\text{eff}} - i\Omega)^2} (1 - e^{(i\Omega - \gamma_{\text{eff}})t_2}) (1 - e^{(i\Omega - \gamma_{\text{eff}})(t_1 - t_2)}) \\ & \left. - \frac{\lambda^2 \omega_{if} \omega_i}{(\gamma_{\text{eff}} + i\Omega)^2} (1 - e^{-(i\Omega + \gamma_{\text{eff}})t_2}) (1 - e^{-(i\Omega + \gamma_{\text{eff}})(t_1 - t_2)}) \right). \end{aligned} \quad (4.180)$$

Here

$$\omega_{i\alpha, f\alpha_1} = \omega_{if} + \frac{1}{\hbar} (E_\alpha - E_{\alpha_1}). \quad (4.181)$$

Approximations

When the dissipation is fast, i.e., the dissipation time is much less than the period of the oscillator, we have $\Omega \ll \gamma_{\text{eff}}$. Then

$$\begin{aligned} \chi_{i\alpha, f\alpha_1}(0, 0; t_1, t_2) = & \exp(-i\omega_{i\alpha, f\alpha_1}(t_1 - t_2)) \\ & \times \exp \left(-(1 + 2\bar{n}(T)) \frac{\lambda^2 \omega_{if}^2}{\gamma_{\text{eff}}} \left(t_1 - t_2 + \frac{1}{\gamma_{\text{eff}}} (e^{-\gamma_{\text{eff}}(t_1 - t_2)} - 1) \right) \right). \end{aligned}$$

The probability of the jump from the level $|i\alpha\rangle$ to the level $|f\alpha_1\rangle$ during the measurement according to Eq. (4.178) is

$$\begin{aligned} W(i\alpha \rightarrow f\alpha_1, t) = & \frac{2t}{\hbar^2} |V_{i\alpha, f\alpha_1}|^2 \text{Re} \int_0^t du \left(1 - \frac{u}{t} \right) e^{i\omega_{f\alpha_1, i\alpha} u} \\ & \times \exp \left(-\frac{(1 + 2\bar{n}(T)) \lambda^2 \omega_{if}^2}{\gamma_{\text{eff}}} \left(u + \frac{1}{\gamma_{\text{eff}}} (e^{-\gamma_{\text{eff}} u} - 1) \right) \right). \end{aligned} \quad (4.182)$$

We introduce the function

$$\Phi_{f\alpha_1, i\alpha}(t) = |V_{i\alpha, f\alpha_1}|^2 \exp \left(\frac{i}{\hbar} [E_1(f, \alpha_1) - E_1(i, \alpha)] t \right) \quad (4.183)$$

and the Fourier transformation of $\Phi_{f\alpha_1, i\alpha}(t)$

$$G_{f\alpha_1, i\alpha}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \Phi_{f\alpha_1, i\alpha}(t) \exp(-i\omega t). \quad (4.184)$$

Then we can rewrite Eq. (4.182) in the form

$$W(i\alpha \rightarrow f\alpha_1, t) = \frac{2\pi t}{\hbar^2} \int_{-\infty}^{\infty} d\omega G_{f\alpha_1, i\alpha}(\omega) P_{if}(\omega), \quad (4.185)$$

where

$$P_{if}(\omega) = \frac{1}{\pi} \operatorname{Re} \int_0^t du \left(1 - \frac{u}{t}\right) \exp(i(\omega - \omega_{if})u) \times \exp\left(-\frac{(1 + 2\bar{n}(T))\lambda^2\omega_{if}^2}{\gamma_{\text{eff}}}\left(u + \frac{1}{\gamma_{\text{eff}}}(e^{-\gamma_{\text{eff}}u} - 1)\right)\right). \quad (4.186)$$

The equation (4.185) is of the form, obtained by Kofman and Kurizki [26], assuming the ideal instantaneous projections. The function $P(\omega)_{if}$ is the measurement-modified shape of the spectral line (Refs. [26, 47, 60]). Here we have shown that Eq. (4.185) can be derived from more realistic model as well. The assumption that dissipation is fast, $\Omega \ll \gamma_{\text{eff}}$ is crucial. Without this assumption the jump probability cannot have the form of Eq. (4.185), since then $\chi_{i\alpha, f\alpha_1}(0, 0; t_1, t_2)$ depends not only on the difference $t_1 - t_2$ but also on t_2 .

When λ is big then to the integral in Eq. (4.182) contribute only small values of u and we can expand the exponent $\exp(-\gamma_{\text{eff}}u)$ into Taylor series keeping the first three terms only. We obtain the jump rate

$$R(i\alpha \rightarrow f\alpha_1) \approx \frac{2}{\hbar^2} |V_{i\alpha, f\alpha_1}|^2 \operatorname{Re} \int_0^\infty du \exp\left(i\omega_{f\alpha_1, i\alpha}u - \frac{1}{2}(1 + 2\bar{n}(T))\lambda^2\omega_{if}^2u^2\right)$$

or

$$R(i\alpha \rightarrow f\alpha_1) \approx \frac{2|V_{i\alpha, f\alpha_1}|^2}{\hbar^2\lambda|\omega_{if}|} \sqrt{\frac{\pi}{2(1 + 2\bar{n}(T))}} \quad (4.187)$$

The obtained decay rate inversely proportional to the measurement strength λ . The measurement strength appears in the equations multiplied by $\sqrt{1 + 2\bar{n}(T)}$, therefore, the effect of the measurement increases as the temperature of the detector grows.

4.7 Quantum trajectory method for the quantum Zeno and anti-Zeno effects

The density matrix analysis assumes that the experiment is performed on a large number of systems. An alternative to the density matrix analysis are stochastic simulation methods [68–72]. Various stochastic simulation methods describe quantum trajectories for the states of the system subjected to random quantum jumps. Using stochastic methods one can examine the behaviour of individual trajectories, therefore such methods provide the description of the experiment on a single system in a more direct way. The results for the ensemble are obtained by repeating the stochastic simulations several times and calculating the average.

Stochastic simulations of the quantum Zeno effect experiment were performed in Ref. [17]. In this Section we describe the evolution of the detector interacting with the environment using the quantum jump model developed by Carmichael [70]. We use the quantum jump method to describe the evolution of frequently measured systems and to compare the numerical results with the analytically obtained decay rates.

4.7.1 Measurement of the unperturbed system

The measured system is described in section 4.2. In this Section we consider the Hamiltonian \hat{H}_0 with two eigenstates: ground $|g\rangle$ and excited $|e\rangle$. At first we investigate the measurement of the unperturbed system, i.e., the case when $\hat{V} = 0$.

We assume that the Markovian approximation is valid i.e., the evolution of the measured system and the detector depends only on their state at the present time. The master equation for the full density matrix of the detector and the measured system is

$$\frac{\partial}{\partial t}\hat{\rho}(t) = \frac{1}{i\hbar}[\hat{H}_0, \hat{\rho}(t)] + \frac{1}{i\hbar}[\hat{H}_I, \hat{\rho}(t)] + \frac{1}{i\hbar}[\hat{H}_D, \hat{\rho}(t)] + \mathcal{L}_D\hat{\rho}(t), \quad (4.188)$$

where the superoperator \mathcal{L}_D accounts for the interaction of the detector with the environment. We assume that the measurement of the unperturbed system is a quantum non-demolition measurement [54–57]. The measurement of the unperturbed system does not change the state of the measured system when initially the system is in an eigenstate of the Hamiltonian \hat{H}_0 . This can be if $[\hat{H}_0, \hat{H}_I] = 0$.

We introduce the superoperator $\mathcal{L}_{n,m}$ acting only on the density matrix of the detector

$$\mathcal{L}_{n,m}\hat{\rho} = \frac{1}{i\hbar}(\langle n|\hat{H}_I|n\rangle\hat{\rho} - \hat{\rho}\langle m|\hat{H}_I|m\rangle) + \frac{1}{i\hbar}[\hat{H}_D, \hat{\rho}] + \mathcal{L}_D\hat{\rho} \quad (4.189)$$

and the superoperator $\mathcal{S}_{n,m}(t)$ obeying the equation

$$\frac{\partial}{\partial t}\mathcal{S}_{n,m}(t) = \mathcal{L}_{n,m}\mathcal{S}_{n,m}(t), \quad (4.190)$$

with the initial condition $\mathcal{S}_{n,m}(0) = 1$. Then the full density matrix of the detector and the measured system after the measurement is

$$\hat{\rho}(\tau_M) = \mathcal{S}(\tau_M)\hat{\rho}(0) = \sum_{n,m} |n\rangle\langle n|(\rho_S)_{n,m}(0)e^{i\omega_{m,n}\tau_M}\langle m| \otimes \mathcal{S}_{n,m}(\tau_M)\hat{\rho}_D(0), \quad (4.191)$$

where τ_M is the duration of the measurement and

$$\omega_{m,n} = \omega_m - \omega_n \quad (4.192)$$

with $\omega_n = E_n/\hbar$. From Eq. (4.191) it follows that the non-diagonal matrix elements of the density matrix of the system after the measurement $(\rho_S)_{n,m}(\tau_M) \equiv (\rho_S)_{n,m}(0)e^{i\omega_{m,n}\tau_M}$ are multiplied by the quantity

$$F_{n,m}(\tau_M) \equiv \text{Tr}\{\mathcal{S}_{n,m}(\tau_M)\hat{\rho}_D(0)\}. \quad (4.193)$$

Since after the measurement the non-diagonal matrix elements of the density matrix of the measured system should become small (they must vanish in the case of an ideal measurement), $F_{n,m}(\tau_M)$ must be also small when $n \neq m$.

4.7.2 The detector

We take an atom with two energy levels, the excited level $|a\rangle$ and the ground level $|b\rangle$, as the detector. The Hamiltonian of the detecting atom is

$$\hat{H}_D = \frac{\hbar\Omega_D}{2}\hat{\sigma}_z. \quad (4.194)$$

Here $\hbar\Omega_D$ defines the separation between levels $|a\rangle$ and $|b\rangle$, $\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$ are Pauli matrices. The interaction Hamiltonian \hat{H}_I we take as

$$\hat{H}_I = \hbar\lambda|g\rangle\langle g|(\hat{\sigma}_+ + \hat{\sigma}_-), \quad (4.195)$$

where $\hat{\sigma}_\pm = \frac{1}{2}(\hat{\sigma}_x \pm i\hat{\sigma}_y)$. The parameter λ describes the strength of the coupling with the detector. The detecting atom interacts with the electromagnetic field. The interaction of the atom with the field is described by the term

$$\mathcal{L}_D\hat{\rho}_D = -\frac{\Gamma}{2}(\hat{\sigma}_+\hat{\sigma}_-\hat{\rho}_D - 2\hat{\sigma}_-\hat{\rho}_D\hat{\sigma}_+ + \hat{\rho}_D\hat{\sigma}_+\hat{\sigma}_-), \quad (4.196)$$

where Γ is the atomic decay rate.

At the equilibrium, when there is no interaction with the measured system, the density matrix of the detector is $\hat{\rho}_D(0) = |b\rangle\langle b|$.

Duration of measurement

We can estimate the characteristic duration of one measurement τ_M as the time during which the non-diagonal matrix elements of the measured system become negligible. Therefore, in order to estimate the duration of the measurement τ_M , we need to calculate the quantity

$$F_{e,g}(t) = \text{Tr}\{\mathcal{S}_{e,g}(t)\hat{\rho}_D(0)\}.$$

We will solve the equation

$$\frac{\partial}{\partial t}\hat{\rho}_D = \frac{1}{i\hbar}[\hat{H}_D, \hat{\rho}_D] + \mathcal{L}_{e,g}\hat{\rho}_D. \quad (4.197)$$

For the matrix elements of the density matrix of the detector we have the equations

$$\frac{\partial}{\partial t}\rho_{aa} = i\lambda\rho_{ab} - \Gamma\rho_{aa}, \quad (4.198)$$

$$\frac{\partial}{\partial t}\rho_{bb} = i\lambda\rho_{ba} + \Gamma\rho_{aa}, \quad (4.199)$$

$$\frac{\partial}{\partial t}\rho_{ab} = -i\Omega_D\rho_{ab} + i\lambda\rho_{aa} - \frac{1}{2}\Gamma\rho_{ab}, \quad (4.200)$$

$$\frac{\partial}{\partial t}\rho_{ba} = i\Omega_D\rho_{ba} + i\lambda\rho_{bb} - \frac{1}{2}\Gamma\rho_{ba}. \quad (4.201)$$

with the initial conditions $\rho_{ab}(0) = \rho_{ba}(0) = \rho_{aa}(0) = 0$, $\rho_{bb}(0) = 1$.

Atom can act as an effective detector when the decay rate Γ of the excited state $|a\rangle$ is large. In such a situation we can obtain approximate solution assuming that ρ_{ba} and ρ_{ab}

are small and ρ_{bb} changes slowly. Then the approximate equation for the matrix element ρ_{ba} is

$$\frac{\partial}{\partial t}\rho_{ba} = i\lambda\rho_{bb}(0) - \frac{1}{2}\Gamma\rho_{ba} \quad (4.202)$$

with the solution

$$\rho_{ba}(t) = 2i\frac{\lambda}{\Gamma}(1 - e^{-\frac{1}{2}\Gamma t}). \quad (4.203)$$

Substituting this solution into the equation for the matrix element ρ_{bb} we get

$$\frac{\partial}{\partial t}\rho_{bb} = -2\frac{\lambda^2}{\Gamma}(1 - e^{-\frac{1}{2}\Gamma t}). \quad (4.204)$$

Taking the term linear in t we obtain the solution

$$\rho_{bb}(t) \approx 1 - 2\frac{\lambda^2}{\Gamma}t \approx e^{-2\frac{\lambda^2}{\Gamma}t}. \quad (4.205)$$

Since $\rho_{aa}(t) = 0$, using Eq. (4.205) we have

$$F_{e,g}(t) = \rho_{bb}(t) \approx \exp\left(-\frac{t}{\tau_M}\right)$$

where

$$\tau_M = \frac{\Gamma}{2\lambda^2} \quad (4.206)$$

is the characteristic duration of the measurement. This estimate is justified comparing with the exact solution of the equations. We get that the measurement duration is shorter for bigger coupling strength λ .

4.7.3 Stochastic methods

The density matrix approach describes the evolution of a large ensemble of independent systems. The observed signal allows us to generate an inferred quantum evolution conditioned by a particular observed record [70]. This gives basis of the quantum jump models. In such models the quantum trajectory is calculated by integrating the time-dependent Schrödinger equation using a non-Hermitian effective Hamiltonian. Incoherent processes such as spontaneous emission are incorporated as random quantum jumps that cause a collapse of the wave function to a single state. Averaging over many realizations of the trajectory reproduces the ensemble results.

The theory of quantum trajectories has been developed by many authors [68–70, 73–76]. Quantum trajectories were used to model continuously monitored open systems [70, 73, 74], in the numerical calculations for the study of dissipative processes [68, 76], and in relation to quantum measurement theory [69, 75].

We assume that the Markovian approximation is valid. The dynamics of the total system consisting of the measured system and the detector is described by a master equation

$$\frac{\partial}{\partial t}\hat{\rho}(t) = \mathcal{M}\hat{\rho}, \quad (4.207)$$

where \mathcal{M} is the superoperator describing the time evolution. The superoperator \mathcal{M} can be separated into two parts

$$\mathcal{M} = \mathcal{L} + \mathcal{J}. \quad (4.208)$$

The part \mathcal{J} is interpreted as describing quantum jumps, \mathcal{L} describes the jump-free evolution. After a short time interval Δt the density matrix is

$$\hat{\rho}(t + \Delta t) = \hat{\rho}(t) + \mathcal{L}\hat{\rho}(t)\Delta t + \mathcal{J}\hat{\rho}(t)\Delta t. \quad (4.209)$$

Since Eq. (4.207) should preserve the trace of the density matrix, we have the equality

$$\text{Tr}\{\mathcal{L}\hat{\rho}(t)\} + \text{Tr}\{\mathcal{J}\hat{\rho}(t)\} = 0. \quad (4.210)$$

Using Eq. (4.210) equation (4.209) can be rewritten in the form

$$\hat{\rho}(t + \Delta t) = \frac{\hat{\rho}(t) + \mathcal{L}\hat{\rho}(t)\Delta t}{1 + \text{Tr}\{\mathcal{L}\hat{\rho}(t)\}\Delta t} (1 - \text{Tr}\{\mathcal{J}\hat{\rho}(t)\}\Delta t) + \frac{\mathcal{J}\hat{\rho}(t)}{\text{Tr}\{\mathcal{J}\hat{\rho}(t)\}} \text{Tr}\{\mathcal{J}\hat{\rho}(t)\}\Delta t. \quad (4.211)$$

This equation can be interpreted in the following way: during the time interval Δt two possibilities can occur. Either after time Δt the density matrix is equal to conditional density matrix

$$\hat{\rho}_{\text{jump}}(t + \Delta t) = \frac{\mathcal{J}\hat{\rho}(t)}{\text{Tr}\{\mathcal{J}\hat{\rho}(t)\}} \quad (4.212)$$

with the probability

$$p_{\text{jump}}(t) = \text{Tr}\{\mathcal{J}\hat{\rho}(t)\}\Delta t \quad (4.213)$$

or to the density matrix

$$\hat{\rho}_{\text{no-jump}}(t + \Delta t) = \frac{\hat{\rho}(t) + \mathcal{L}\Delta t\hat{\rho}(t)}{1 + \text{Tr}\{\mathcal{L}\hat{\rho}(t)\}\Delta t} \quad (4.214)$$

with the probability $1 - p_{\text{jump}}(t)$. Thus the equation (4.207) can be replaced by the stochastic process.

Here we assume that the superoperators \mathcal{L} and \mathcal{J} have the form

$$\mathcal{L}\hat{\rho} = \frac{1}{i\hbar}(\hat{H}_{\text{eff}}\hat{\rho} - \hat{\rho}\hat{H}_{\text{eff}}^\dagger), \quad (4.215)$$

$$\mathcal{J}\hat{\rho} = \hat{C}\hat{\rho}\hat{C}^\dagger. \quad (4.216)$$

The operators \hat{H}_{eff} and \hat{C} are non-Hermitian in general. If the superoperators \mathcal{L} and \mathcal{J} have the form given in Eqs. (4.215), (4.216) and the density matrix at the time t factorizes as $\hat{\rho}(t) = |\Psi(t)\rangle\langle\Psi(t)|$ then after time interval Δt the density matrices $\hat{\rho}_{\text{jump}}(t + \Delta t)$ and $\hat{\rho}_{\text{no-jump}}(t + \Delta t)$ factorize also. Therefore, equation for density matrix (4.211) can be replaced by the corresponding equation for the state vectors. The state vector after time Δt in which a jump is recorded is given by

$$|\Psi_{\text{jump}}(t + \Delta t)\rangle = \frac{1}{\sqrt{\langle\Psi(t)|\hat{C}^\dagger\hat{C}|\Psi(t)\rangle}}\hat{C}|\Psi(t)\rangle. \quad (4.217)$$

The probability of a jump occurring in the time interval Δt is

$$p_{\text{jump}}(t) = \langle \Psi(t) | \hat{C}^\dagger \hat{C} | \Psi(t) \rangle \Delta t. \quad (4.218)$$

If no jump occurs, the state vector evolves according to the non-Hermitian Hamiltonian \hat{H}_{eff} ,

$$|\Psi_{\text{jump}}(t + \Delta t)\rangle = \frac{1}{\sqrt{\langle \Psi(t) | \left(1 + \frac{i}{\hbar}(\hat{H}_{\text{eff}}^\dagger - \hat{H}_{\text{eff}})\Delta t\right) | \Psi(t) \rangle}} \left(1 - \frac{i}{\hbar} \hat{H}_{\text{eff}} \Delta t\right) |\Psi(t)\rangle. \quad (4.219)$$

Numerical simulation takes place over discrete time with time step Δt . When the wavefunction $|\Psi(t_n)\rangle$ is given, the wavefunction $|\Psi(t_{n+1})\rangle$ is determined by the following algorithm [70]:

1. evaluate the collapse probability $p_{\text{jump}}(t_n)$ according to Eq. (4.218)
2. generate a random number r_n distributed uniformly on the interval $[0, 1]$
3. compare $p_{\text{jump}}(t_n)$ with r_n and calculate $|\Psi_c(t_{n+1})\rangle$ according to the rule

$$\begin{aligned} |\Psi(t_{n+1})\rangle &\sim \hat{C} |\Psi_c(t_n)\rangle, & p_{\text{jump}}(t_n) < r_n, \\ |\Psi(t_{n+1})\rangle &\sim \exp\left(-\frac{i}{\hbar} \hat{H}_{\text{eff}} \Delta t\right) |\Psi(t_n)\rangle, & p_{\text{jump}}(t_n) > r_n. \end{aligned}$$

We can approximate the second case as

$$|\Psi(t_{n+1})\rangle \sim \left(1 - \frac{i}{\hbar} \hat{H}_{\text{eff}} \Delta t\right) |\Psi(t_n)\rangle.$$

4.7.4 Stochastic simulation of the detector

At first we consider the measurement of the unperturbed system and take the perturbation $\hat{V} = 0$. The measured system is an atom with the states $|g\rangle$ and $|e\rangle$. The Hamiltonian of the measured atom is

$$\hat{H}_0 = \hbar\omega_{eg}|e\rangle\langle e|, \quad (4.220)$$

where $\hbar\omega_{eg}$ is the energy of the excited level.

The stochastic methods described in Sec. 4.7.3 were used to perform the numerical simulations of the measurement process. Using the equation (4.196) we take the operator \hat{C} in Eq. (4.216) describing jumps in the form

$$\hat{C} = \sqrt{\Gamma} \hat{\sigma}_- \quad (4.221)$$

and the effective Hamiltonian in Eq. (4.215) as

$$\hat{H}_{\text{eff}} = \hat{H}_0 + \hat{H}_D + \hat{H}_I - i\hbar \frac{\Gamma}{2} \hat{\sigma}_+ \hat{\sigma}_-. \quad (4.222)$$

The wavefunction of the measured system and the detector is expressed in the basis of the eigenfunctions of the Hamiltonians of the measured system and the detector

$$|\Psi\rangle = c_{ea}|e\rangle|a\rangle + c_{eb}|e\rangle|b\rangle + c_{ga}|g\rangle|a\rangle + c_{gb}|g\rangle|b\rangle. \quad (4.223)$$

The effective Hamiltonian produces the following equations for the coefficients of the wave function $|\Psi\rangle$

$$\dot{c}_{ea} = -i \left(\omega_{eg} + \frac{\Omega_D}{2} - i\frac{\Gamma}{2} \right) c_{ea}, \quad (4.224)$$

$$\dot{c}_{eb} = -i \left(\omega_{eg} - \frac{\Omega_D}{2} \right) c_{eb}, \quad (4.225)$$

$$\dot{c}_{ga} = -i\lambda c_{gb} - i\frac{\Omega_D}{2}c_{ga} - \frac{\Gamma}{2}c_{ga}, \quad (4.226)$$

$$\dot{c}_{gb} = -i\lambda c_{ga} + i\frac{\Omega_D}{2}c_{gb}. \quad (4.227)$$

Equations (4.224)–(4.227) are used in the numerical simulations to describe the evolution between the jumps. After the jump in the detecting atom the unnormalized wavefunction is

$$\hat{C}|\Psi\rangle = \sqrt{\Gamma}(c_{ea}|e\rangle|b\rangle + c_{ga}|g\rangle|b\rangle). \quad (4.228)$$

The jump occurs with the probability p_{jump} obtained from Eq. (4.218),

$$p_{\text{jump}} = \Gamma \Delta t \frac{|c_{ea}|^2 + |c_{ga}|^2}{|c_{ea}|^2 + |c_{ga}|^2 + |c_{eb}|^2 + |c_{gb}|^2}. \quad (4.229)$$

For numerical simulation we take the measured system in an initial superposition state $\frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$. The typical quantum trajectories of the detector are shown in Fig. 4.4. There are two kinds of trajectories corresponding to the collapse of the measured system to the excited or the ground states. The trajectories corresponding to the collapse of the measured system to the ground state show the repeated jumps. The mean interval between jumps, obtained from the numerical simulation is of the same order of magnitude as $\tau_M = 5$ according to Eq. (4.206). After averaging over the realizations the probability for the detector to be in the excited state is shown in Fig. 4.5. The figure shows that this probability reaches the stationary value. The time dependency of the non-diagonal matrix elements of the density matrix of the measured system is shown in Fig. 4.6. The figure shows a good agreement between the results of numerical calculations and the exponential decay with the characteristic time estimated from Eq. (4.206).

4.7.5 Frequently measured perturbed two level system

We consider an atom interacting with the classical external electromagnetic field as the measured system. Interaction of the atom with the field is described by the operator

$$\hat{V} = -\hbar\Omega_R(|e\rangle\langle g| + |g\rangle\langle e|) \cos \Omega t, \quad (4.230)$$

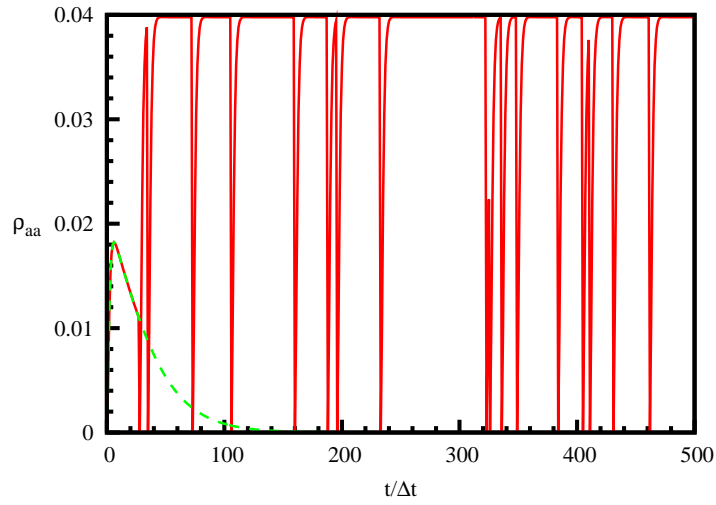


Figure 4.4: Typical quantum trajectories of the detector. Figure shows the probability ρ_{aa} of being in the excited level of the detector. The solid line corresponds to the case when the measured system collapses to the ground state and the dashed line corresponds to the case when the measured system collapses to the excited state. The parameters used for numerical calculation are $\Delta t = 0.1$, $\Gamma = 10$, $\lambda = 1$, and $\Omega_D = 1$.

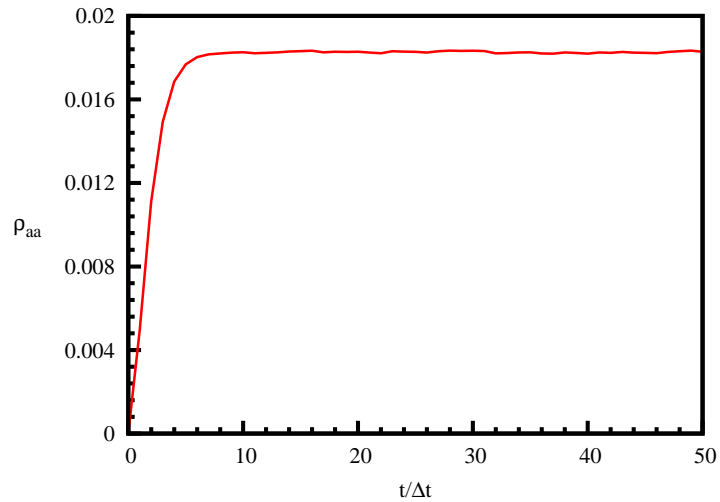


Figure 4.5: Probability for the detector to be in the excited state, after performing an ensemble average over 1000 trajectories. The parameters used are the same as in Fig. 4.4.

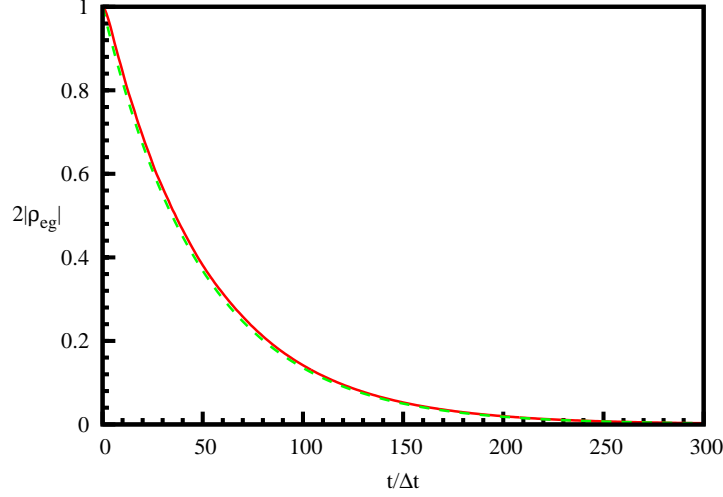


Figure 4.6: Non-diagonal matrix elements of the density matrix of the measured system. The solid line corresponds to the numerical calculations and the dashed line corresponds to the exponential decay with the characteristic time given by Eq. (4.206). The parameters used are the same as in Fig. 4.4.

where Ω is the frequency of the field and Ω_R is the Rabi frequency. In the interaction representation and using the rotating-wave approximation the perturbation \hat{V} is

$$\tilde{V}(t) = -\hbar \frac{\Omega_R}{2} (e^{i\Delta\omega t} |e\rangle\langle g| + e^{-i\Delta\omega t} |g\rangle\langle e|), \quad (4.231)$$

where

$$\Delta\omega = \omega_{eg} - \Omega \quad (4.232)$$

is the detuning. Here $\hbar\omega_{eg} = \hbar\omega_e - \hbar\omega_g$ is the energy difference between the excited and the ground levels of the measured atom.

If the measurements are not performed, the atom exhibits Rabi oscillations with the frequency Ω_R . If the measured atom is initially in the state $|g\rangle$, the time dependence of the coefficient c_g of the wavefunction $|\Psi\rangle = c_e|e\rangle + c_g|g\rangle$ is

$$c_g(t) = e^{-\frac{1}{2}it\Delta\omega} \left(\cos\left(\frac{1}{2}t\sqrt{\Delta\omega^2 + \Omega_R^2}\right) + i \frac{\Delta\omega}{\sqrt{\Delta\omega^2 + \Omega_R^2}} \sin\left(\frac{1}{2}t\sqrt{\Delta\omega^2 + \Omega_R^2}\right) \right). \quad (4.233)$$

In particular, if the detuning $\Delta\omega$ is zero, we have

$$c_g(t) = c_g(0) \cos\left(\frac{\Omega_R}{2}t\right). \quad (4.234)$$

When the measured atom interacts with the detector, we take the wavefunction of the measured system and of the detector as in Eq. (4.223). In the interaction representation the equations for the coefficients, when the evolution is governed by the effective

Hamiltonian \hat{H}_{eff} , defined by Eq. (4.215), are

$$\dot{c}_{ea} = i\frac{\Omega_R}{2}e^{it\Delta\omega}c_{ga} - i\frac{\Omega_D}{2}c_{ea} - \frac{\Gamma}{2}c_{ea} \quad (4.235)$$

$$\dot{c}_{eb} = i\frac{\Omega_R}{2}e^{it\Delta\omega}c_{gb} + i\frac{\Omega_D}{2}c_{eb} \quad (4.236)$$

$$\dot{c}_{ga} = i\frac{\Omega_R}{2}e^{-it\Delta\omega}c_{ea} - i\lambda c_{gb} - i\frac{\Omega_D}{2}c_{ga} - \frac{\Gamma}{2}c_{ga} \quad (4.237)$$

$$\dot{c}_{gb} = i\frac{\Omega_R}{2}e^{-it\Delta\omega}c_{eb} - i\lambda c_{ga} + i\frac{\Omega_D}{2}c_{gb} \quad (4.238)$$

The evolution of the measured atom significantly differs from the Rabi oscillations. We are interested in the case when the duration of the measurement τ_M is much shorter than the period of Rabi oscillations $2\pi/\Omega_R$. In such a situation the non diagonal matrix elements of the density matrix of the measured system remain small and the time evolution of the diagonal matrix elements can be approximately described by the rate equations

$$\frac{d}{dt}\rho_{gg} = \Gamma_{e\rightarrow g}\rho_{ee}(t) - \Gamma_{g\rightarrow e}\rho_{gg}(t), \quad (4.239)$$

$$\frac{d}{dt}\rho_{ee} = \Gamma_{g\rightarrow e}\rho_{gg}(t) - \Gamma_{e\rightarrow g}\rho_{ee}(t). \quad (4.240)$$

If the measured atom is initially in the state $|g\rangle$, the solution of Eqs. (4.239) and (4.240) is

$$\rho_{gg}(t) = \frac{\Gamma_{e\rightarrow g} + \Gamma_{g\rightarrow e}e^{-(\Gamma_{e\rightarrow g} + \Gamma_{g\rightarrow e})t}}{\Gamma_{e\rightarrow g} + \Gamma_{g\rightarrow e}} = \frac{1}{2}(1 + e^{-2\Gamma_{g\rightarrow e}t}). \quad (4.241)$$

We can estimate the rates $\Gamma_{e\rightarrow g}$ and $\Gamma_{g\rightarrow e}$ using equations from Ref. [77], i.e.,

$$\Gamma_{e\rightarrow g} = 2\pi \int_{-\infty}^{\infty} G(\omega)P_{eg}(\omega)d\omega, \quad (4.242)$$

$$\Gamma_{g\rightarrow e} = 2\pi \int_{-\infty}^{\infty} G(\omega)P_{ge}(\omega)d\omega, \quad (4.243)$$

where

$$P_{eg}(\omega) = \frac{1}{\pi} \text{Re} \int_0^{\infty} F_{eg}(\tau)e^{i(\omega - \omega_{eg})\tau}d\tau \quad (4.244)$$

$$P_{ge}(\omega) = \frac{1}{\pi} \text{Re} \int_0^{\infty} F_{ge}(\tau)e^{i(\omega + \omega_{eg})\tau}d\tau \quad (4.245)$$

and

$$G(\omega) = \left(\frac{\Omega_R}{2}\right)^2 [\delta(\omega - \omega_{eg} + \Delta\omega) + \delta(\omega + \omega_{eg} - \Delta\omega)]. \quad (4.246)$$

Here the expression for $G(\omega)$ is derived using Eq. (4.231). In contrast to Ref. [77] in the expression for $P(\omega)$ we extended the range of the integration to the infinity since $F_{eg}(\tau)$ naturally limits the duration of the measurement. Expressions, analogous to (4.242), were obtained in Refs. [26, 60], as well.

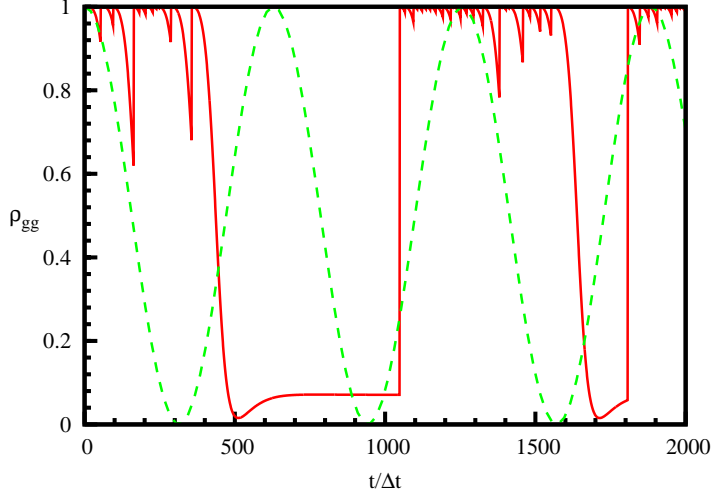


Figure 4.7: Typical quantum trajectory of the measured two level system (solid line). Figure shows the probability ρ_{gg} for the measured atom to be in the ground level. The dashed line shows the Rabi oscillations in the measurement-free evolution. The detuning $\Delta\omega$ is zero. The parameters used for the numerical calculation are $\Delta t = 0.1$, $\Gamma = 10$, $\lambda = 1$, $\Omega_D = 1$, and $\Omega_R = 0.1$.

Using Eqs. (4.242)–(4.246) we can estimate the transition rates as

$$\Gamma_{e \rightarrow g} \approx \Gamma_{g \rightarrow e} \approx \frac{\Omega_R^2}{2} \operatorname{Re} \int_0^\infty F_{eg}(\tau) e^{-i\tau\Delta\omega} d\tau = \frac{\Omega_R^2}{2} \int_0^\infty e^{-\frac{\tau}{\tau_M} - i\tau\Delta\omega} d\tau = \frac{\Omega_R^2}{2} \frac{\tau_M}{1 + (\tau_M\Delta\omega)^2}. \quad (4.247)$$

Here we used the expression $\exp(-\tau/\tau_M)$ for $F_{eg}(\tau)$. When the detuning $\Delta\omega$ is zero the transition rates are

$$\Gamma_{e \rightarrow g} \approx \Gamma_{g \rightarrow e} \approx \frac{\Omega_R^2 \tau_M}{2}. \quad (4.248)$$

The transition rates are smaller for the shorter measurements. This is a manifestation of the quantum Zeno effect.

For numerical simulation we take the measured system in an initial state $|g\rangle$. The typical quantum trajectory of the measured system is shown in Fig. 4.7. The behaviour of the measured system strongly differs from the measurement-free evolution. The measurement-free system oscillates with the Rabi frequency, while the measured system stays in one of the levels and suddenly jumps to the other. The probability for the measured atom to be in the ground state calculated after averaging over the realizations is shown in Fig. 4.8. The figure shows that this probability exhibits almost the exponential decay and after some time reaches the stationary value close to $1/2$. The figure shows a good agreement between the results of the numerical calculations and the estimate (4.241).

When the detuning $\Delta\omega$ is not zero, the frequently measured two level system can exhibit the anti-Zeno effect. This is pointed out in Ref. [78]. For the case of nonzero detuning the probability that the atom is in the ground state is shown in Fig. 4.9. The figure shows that the probability for the atom to be in the initial (ground) state is smaller, and, consequently, to be in the excited state is greater when the atom is measured. This is the manifestation of the quantum anti-Zeno effect in the two level system.

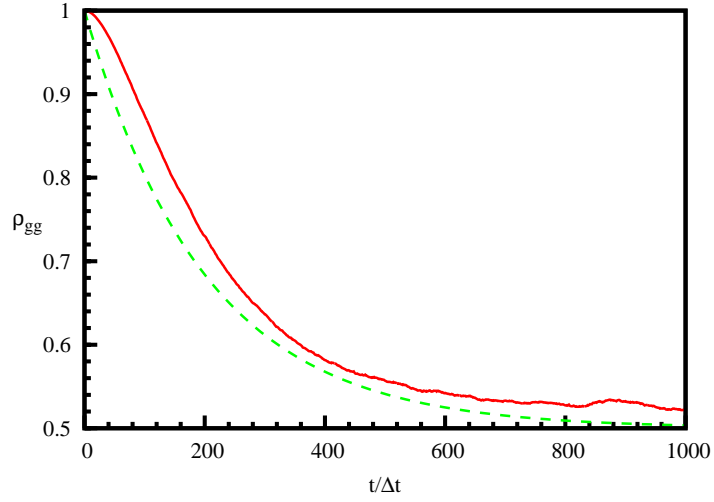


Figure 4.8: Probability for the measured atom to be in the ground level, after performing an ensemble average over 1000 trajectories. The solid line shows the results of the numerical calculations, the dashed line shows the approximation according to Eq. (4.241). The parameters used are the same as in Fig. 4.7.

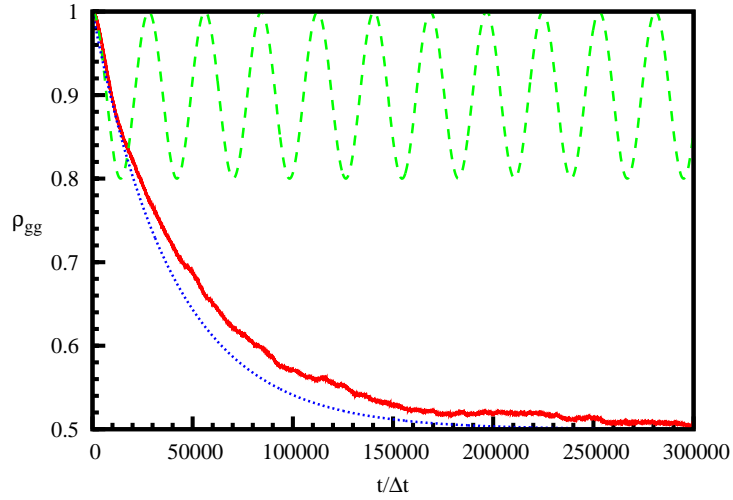


Figure 4.9: Probability for the atom with the nonzero detuning to be in the ground level. The solid line shows the results of the numerical simulation. The dotted line shows the approximation according to Eq. (4.241), using the decay rate from Eq. (4.247). The dashed line shows the evolution of a not measured system. The parameters used for numerical calculation are $\Delta t = 0.001$, $\Gamma = 10$, $\lambda = 1$, $\Omega_D = 1$, $\Omega_R = 0.1$, and $\Delta\omega = 0.2$.

4.7.6 Decaying system

We model the decaying system as two level system interacting with the reservoir consisting of many levels. The full Hamiltonian of the system is

$$\hat{H} = \hat{H}_0 + \hat{H}_1 + \hat{V}, \quad (4.249)$$

where

$$\hat{H}_0 = \hbar\omega_{eg}|e\rangle\langle e| \quad (4.250)$$

is the Hamiltonian of the two level system,

$$\hat{H}_1 = \sum_k \hbar\omega_k|k\rangle\langle k|. \quad (4.251)$$

is the Hamiltonian of the reservoir, and

$$\hat{V} = \hbar \sum_k (g(k)|e\rangle\langle k| + g(k)^*|k\rangle\langle e|) \quad (4.252)$$

describes the interaction of the system with the reservoir, with $g(k)$ being the strength of the interaction with reservoir mode k . In the interaction representation the perturbation V has the form

$$\tilde{V}(t) = e^{\frac{i}{\hbar}(\hat{H}_0 + \hat{H}_1)t} \hat{V} e^{-\frac{i}{\hbar}(\hat{H}_0 + \hat{H}_1)t} = \hbar \sum_k (g(k)e^{i(\omega_{eg} - \omega_k)t} |e\rangle\langle k| + g(k)^* e^{-i(\omega_{eg} - \omega_k)t} |k\rangle\langle e|).$$

The wavefunction of the system in the interaction representation is expressed as

$$|\tilde{\Psi}\rangle = c_e(t)|e\rangle|0\rangle + \sum_k c_k|g\rangle|k\rangle. \quad (4.253)$$

One can then obtain from the Schrödinger equation the following equations for the coefficients

$$\dot{c}_e = -i \sum_k g(k)e^{i(\omega_{eg} - \omega_k)t} c_k, \quad (4.254)$$

$$\dot{c}_k = -ig(k)^* e^{-i(\omega_{eg} - \omega_k)t} c_e. \quad (4.255)$$

The initial condition is $|\Psi\rangle = |e\rangle|0\rangle$. Formally integrating the equation (4.255) we obtain the expression

$$c_k = -ig(k)^* \int_0^t e^{-i(\omega_{eg} - \omega_k)t'} c_e(t') dt'. \quad (4.256)$$

Inserting Eq. (4.256) into Eq. (4.254), we obtain the exact integro-differential equation

$$\frac{d}{dt} c_e = - \int_0^t dt' \sum_k |g(k)|^2 e^{i(\omega_{eg} - \omega_k)(t-t')} c_e(t'). \quad (4.257)$$

The sum over k may be replaced by an integral

$$\sum_k \rightarrow \int d\omega_k \rho(\omega_k)$$

with $\rho(\omega_k)$ being the density of states in the reservoir. The integration in Eq. (4.257) can be carried out in the Weisskopf-Wigner approximation. We get the equation

$$\frac{d}{dt}c_e = -\frac{\Gamma_{e \rightarrow g}^{(0)}}{2}c_e, \quad (4.258)$$

where the decay rate $\Gamma_{e \rightarrow g}^{(0)}$ is given by the Fermi's Golden Rule:

$$\Gamma_{e \rightarrow g}^{(0)} = 2\pi\rho(\omega_{eg})|g(\omega_{eg})|^2. \quad (4.259)$$

In order to observe the quantum anti-Zeno effect one needs to have sufficiently big derivative of the quantity $\rho(\omega)|g(\omega)|^2$. In such a case the decay rate given by Fermi Golden Rule (4.259) is no longer valid. The corrected decay rate may be obtained solving Eq. (4.257) by the Laplace transform method [79]. The the Laplace transform of the solution of Eq. (4.257) is

$$\tilde{c}_e(z) = \frac{1}{\mathcal{H}(z)}, \quad (4.260)$$

where $\mathcal{H}(z)$ is the resolvent function

$$\mathcal{H}(z) = z + \int \frac{\rho(\omega)|g(\omega)|^2}{z + i(\omega - \omega_{eg})}d\omega. \quad (4.261)$$

In the numerical calculations we take the frequencies of the reservoir ω distributed in the region $[\omega_{eg} - \Lambda, \omega_{eg} + \Lambda]$ with the constant spacing $\Delta\omega$. Therefore, the density of states is constant $\rho(\omega) = 1/\Delta\omega \equiv \rho_0$. The simplest choice of the interaction strength $g(\omega)$ is to make it linearly dependent on ω ,

$$g(\omega) = g_0 \left(1 + \frac{a}{\Lambda}(\omega - \omega_{eg}) \right), \quad (4.262)$$

where a is dimensionless parameter. Using Eq. (4.262) one obtains the expression for the resolvent

$$\mathcal{H}(z) = z + \pi\rho_0g_0^2 \left[1 - \frac{2}{\pi} \arctan \left(\frac{z}{\Lambda} \right) + \left(a^2 \frac{z}{\Lambda} - i2a \right) \left(\frac{2}{\pi} - \frac{z}{\Lambda} + \frac{2}{\pi} \frac{z}{\Lambda} \arctan \left(\frac{z}{\Lambda} \right) \right) \right].$$

The real part of z at which the resolvent $\mathcal{H}(z)$ is equal to zero gives the decay rate. Expanding the resolvent into the series of powers of Λ^{-1} and keeping only the first-order terms we obtain the decay rate

$$\Gamma_{e \rightarrow g}^{(1)} \approx \Gamma_{e \rightarrow g}^{(0)} \left(1 - \frac{\Gamma_{e \rightarrow g}^{(0)}}{\pi\Lambda} (5a^2 - 1) \right). \quad (4.263)$$

We solve Eqs. (4.254) and (4.255) numerically, replacing them with discretized versions with the time step Δt . For calculations we used $N = 1000$ levels in the reservoir. The numerical results for constant interaction strength are $g(k) = g_0$ presented in Fig. 4.10. The figure shows a good agreement between the numerical results and the exponential law according to the Fermi's Golden Rule at intermediate times. At very short time

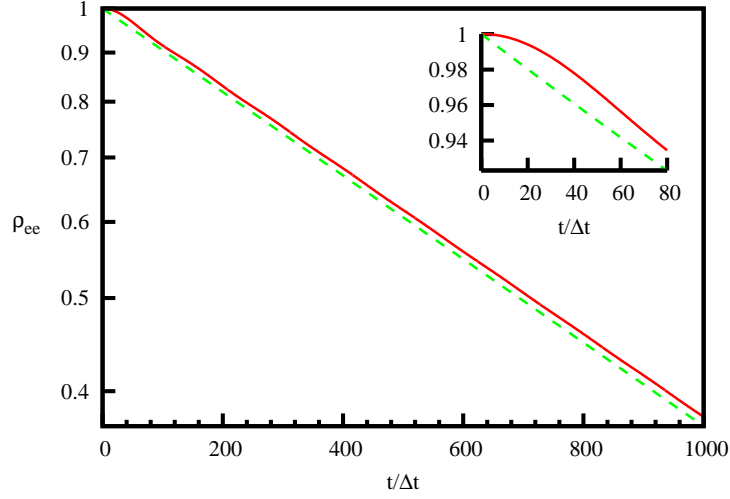


Figure 4.10: Time dependence of the occupation of the exited level of the decaying system. Solid line shows the results of the numerical calculation, dashed line shows the exponential decay according to the Fermi's Golden Rule. The parameters used for the numerical calculation are $\Delta t = 0.1$, $\Delta\omega = 0.001$, $\Lambda = 0.5$, and $g_0 = 0.001262$. For the parameters used the decay rate is $\Gamma_{e \rightarrow g}^{(0)} = 0.01$.

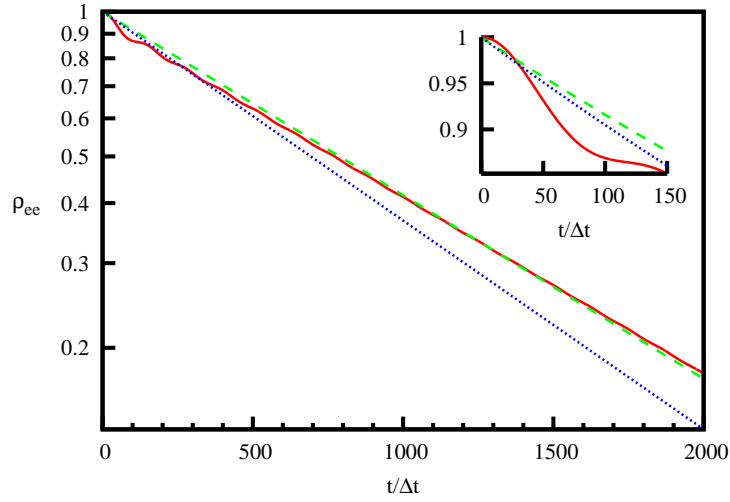


Figure 4.11: Time dependence of the occupation of the exited level of the decaying system when the interaction with the reservoir modes is described by Eq. (4.262). Solid line shows results of numerical calculation, dashed line shows exponential decay using the decay rate from Eq. (4.263), dotted line shows exponential decay according to Fermi's Golden Rule (4.259). In the calculations we used $a = 2$, other parameters are the same as in Fig. 4.10.

the occupation of the excited level exhibits quadratic behaviour, which, for the repeated frequent measurements, may result in the quantum Zeno effect.

Numerical results in the case when the interaction with the reservoir modes is described by Eq. (4.262) with nonzero parameter a are presented in Fig. 4.11. The figure shows good agreement between the numerical results and the exponential decay with the decay rate given by Eq. (4.263) at intermediate times. For very short time we observe the acceleration of the decay due to the interaction with the reservoir. This acceleration for the repeated frequent measurements results in the quantum anti-Zeno effect.

4.7.7 Measurement of the decaying system

In this section we consider the decaying system, described in Sec. 4.7.6 and interacting with the detector. The wavefunction of the measured system and the detector, when the detector interacts with the decaying system in the ground state only, we take in the form

$$|\Psi_c\rangle = c_{ea}|e\rangle|0\rangle|a\rangle + c_{eb}|e\rangle|0\rangle|b\rangle + \sum_k (c_{ka}|g\rangle|k\rangle|a\rangle + c_{kb}|g\rangle|k\rangle|b\rangle). \quad (4.264)$$

The equations for the coefficients, when the evolution is governed by the effective Hamiltonian \hat{H}_{eff} , are

$$\dot{c}_{ea} = -i \sum_k g(k) e^{i(\omega_{eg} - \omega_k)t} c_{ka} - i \frac{\Omega_D}{2} c_{ea} - \frac{\Gamma}{2} c_{ea}, \quad (4.265)$$

$$\dot{c}_{eb} = -i \sum_k g(k) e^{i(\omega_{eg} - \omega_k)t} c_{kb} + i \frac{\Omega_D}{2} c_{eb}, \quad (4.266)$$

$$\dot{c}_{ka} = -ig(k)^* e^{-i(\omega_{eg} - \omega_k)t} c_{ea} - i\lambda c_{kb} - i \frac{\Omega_D}{2} c_{ka} - \frac{\Gamma}{2} c_{ka}, \quad (4.267)$$

$$\dot{c}_{kb} = -ig(k)^* e^{-i(\omega_{eg} - \omega_k)t} c_{eb} - i\lambda c_{ka} + i \frac{\Omega_D}{2} c_{kb}. \quad (4.268)$$

After the jump in the detecting atom the unnormalized wavefunction becomes

$$\hat{C}|\Psi_c\rangle = \sqrt{\Gamma} (c_{ea}|e\rangle|0\rangle|b\rangle + \sum_k c_{ka}|g\rangle|k\rangle|b\rangle). \quad (4.269)$$

According to Ref. [77], the decay rate of the measured system is given by expression (4.242) with

$$G(\omega) = \rho(\omega) |g(\omega)|^2 \quad (4.270)$$

and

$$P(\omega) = \frac{1}{\pi} \text{Re} \int_0^\infty F_{eg}(\tau) e^{i(\omega - \omega_{eg})\tau} d\tau. \quad (4.271)$$

Using $F_{eg}(\tau) = \exp(-\tau/\tau_M)$ we obtain

$$P(\omega) = \frac{1}{\pi} \frac{\tau_M}{1 + (\omega - \omega_{eg})^2 \tau_M^2}. \quad (4.272)$$

In order to obtain the quantum Zeno effect we take $G(\omega)$ as a constant

$$G(\omega) = \frac{\hbar^2 g_0^2}{\Delta\omega}, \quad \omega_{eg} - \Lambda \leq \omega \leq \omega_{eg} + \Lambda. \quad (4.273)$$

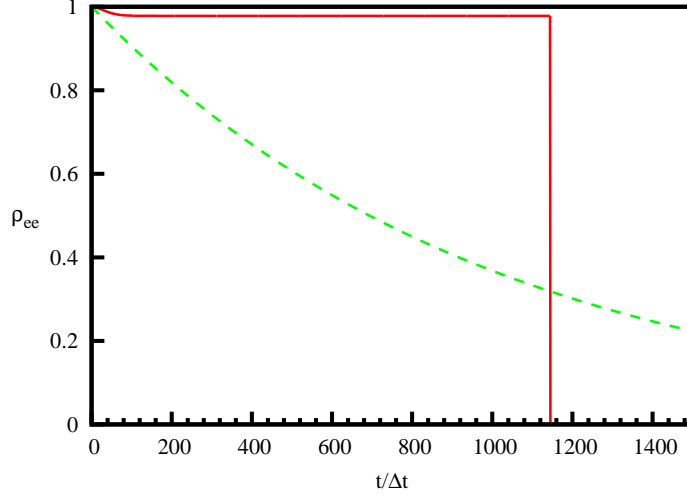


Figure 4.12: Typical quantum trajectory (solid line) of the measured decaying system when the detector interacts with the ground state of the measured system. Figure shows the probability ρ_{ee} that the measured atom is in the excited level. The dashed line shows the exponential decay according to the Fermi's Golden Rule in the measurement-free evolution. The parameters used for the numerical calculation are $\Delta t = 0.1$, $\Gamma = 10$, $\lambda = 1$, $\Omega_D = 1$, $\Delta\omega = 0.001$, $\Lambda = 0.5$, and $g_0 = 0.001262$. For the parameters used the decay rate is $\Gamma_{e \rightarrow g} = 0.01$.

Here $\Delta\omega$ is the spacing between the modes of the reservoir. Using Eq. (4.242) we get the decay rate of the measured decaying system

$$\Gamma_{e \rightarrow g} = \Gamma_{e \rightarrow g}^{(0)} \frac{2}{\pi} \arctan(\Lambda\tau_M). \quad (4.274)$$

When $\Lambda\tau_M$ is big, we obtain the expression

$$\Gamma_{e \rightarrow g} = \Gamma_{e \rightarrow g}^{(0)} \left(1 - \frac{2}{\pi} \frac{1}{\Lambda\tau_M} + \dots \right) \quad (4.275)$$

by expanding Eq. (4.274) into series of the powers of $(\Lambda\tau_M)^{-1}$. The second term in Eq. (4.275) shows that the decay rate decreases with the decreasing duration of the measurement τ_M . This is the manifestation of the quantum Zeno effect.

The results of the numerical simulation are presented in Figs. 4.12 and 4.13. Typical quantum trajectory of the measured decaying system is shown in Fig. 4.12. This trajectory is compatible with the intuitive quantum jump picture: the system stays in the excited state for some time and then suddenly jumps to the ground state. The probability that the measured system stays in the excited state is presented in Fig. 4.13. Figure shows a good agreement between the numerical simulation and the exponential law approximation with the exponent given in Eq. (4.274). Also the quantum Zeno effect is apparent.

When the detector interacts with the excited state of the decaying system the interaction term is

$$\hat{H}_I = \hbar\lambda|e\rangle\langle e|(\hat{\sigma}_+ + \hat{\sigma}_-) \quad (4.276)$$

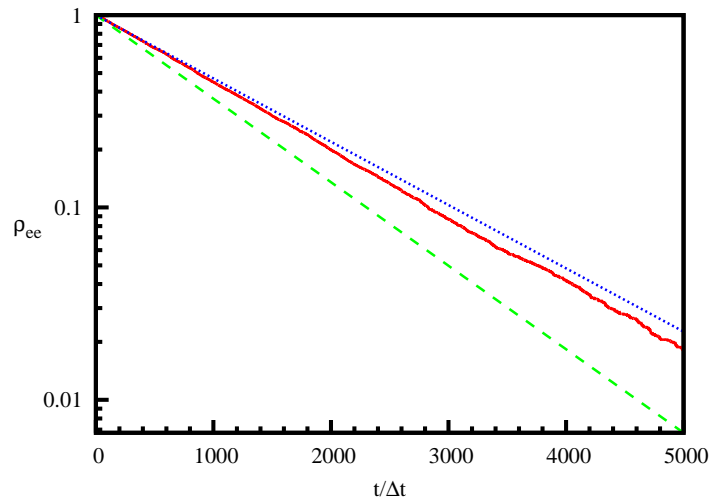


Figure 4.13: Time dependence of the occupation of the excited level of the decaying system. Solid line shows results of numerical calculation, dashed line shows exponential decay according to Fermi's Golden Rule. The dotted line shows approximation according to Eq. (4.274). The parameters used are the same as in Fig. 4.12.

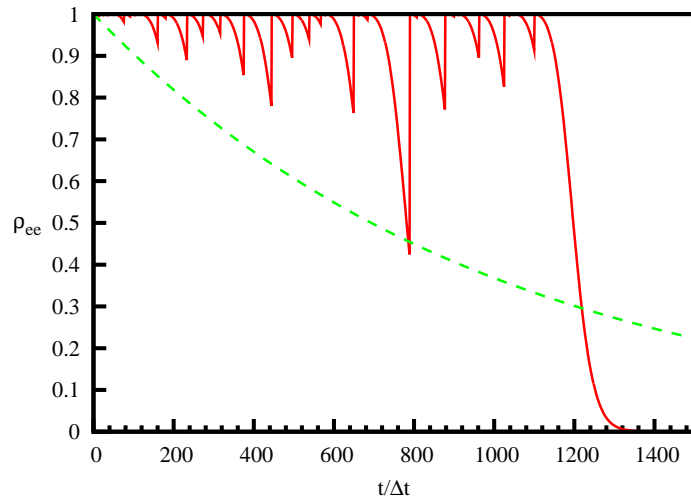


Figure 4.14: Typical quantum trajectory of the measured decaying system (solid line) when the detector interacts with the excited level. Figure shows the probability ρ_{ee} that the measured atom is in the excited level. The dashed line shows exponential decay according to the Fermi's Golden Rule in the measurement-free evolution. The parameters used are the same as in Fig. 4.12.

and the quantum trajectories are different. Typical quantum trajectory is shown in Fig. 4.14. This difference can be explained in the following way: when the detector interacts with the ground state, the interaction effectively begins only after some time, when the probability to find the system in the ground state is sufficiently big. This explains the absence of the collapses at short times in Fig. 4.12. Then, the measurement result after the collapse most likely will be that the system is found in the ground state. When the detector interacts with the excited state of the system, the interaction starts immediately and soon after that the most probable result of the measurement is that the measured system is in the excited state. It should be noted, that the averaged evolution shown in Fig. 4.13 does not depend on the state the detector is interacting.

The model used for the decaying system when $g(\omega) = \text{const}$ does not allow to obtain the quantum anti-Zeno effect, since the conditions for the quantum anti-Zeno effect, presented in Ref. [26], are not satisfied. In order to obtain the quantum anti-Zeno effect we use the interaction with the reservoir modes described by Eq. (4.262). In such a case we have

$$G(\omega) = \frac{\hbar^2 g_0^2}{\Delta\omega} \left(1 + \frac{a}{\Lambda}(\omega - \omega_{eg})\right)^2, \quad \omega_{eg} - \Lambda \leq \omega \leq \omega_{eg} + \Lambda. \quad (4.277)$$

Equation (4.242) in this case does not give correct decay rate of the measured system. In order to estimate the decay rate, we solve the Liouville-von Neumann equation $i\hbar\dot{\rho} = [\hat{H}, \rho]$ for the density matrix of the system with the Hamiltonian (4.249)–(4.252), including additional terms describing decay of the non-diagonal elements with rate $1/\tau_M$, i.e.,

$$\dot{\rho}_{e0,e0} = -i \sum_k (g(k)\rho_{gk,e0} - \rho_{e0,gk}g(k)^*), \quad (4.278)$$

$$\dot{\rho}_{gk,gk'} = -i\omega_{kk'}\rho_{gk,gk'} - i(g(k)^*\rho_{e0,gk'} - \rho_{gk,e0}g(k')), \quad (4.279)$$

$$\dot{\rho}_{e0,gk} = \left(-i(\omega_{eg} - \omega_k) - \frac{1}{\tau_M}\right)\rho_{e0,gk} - i\left(\sum_{k'} g(k')\rho_{gk',gk} - \rho_{e0,e0}g(k)\right), \quad (4.280)$$

$$\dot{\rho}_{gk,e0} = \left(-i(\omega_k - \omega_{eg}) - \frac{1}{\tau_M}\right)\rho_{gk,e0} - i(g(k)^*\rho_{e0,e0} - \sum_{k'} \rho_{gk,gk'}g(k')^*). \quad (4.281)$$

We solve equations (4.278)–(4.281) using the Laplace transform method. Eliminating $\tilde{\rho}_{e0,gk}$ and $\tilde{\rho}_{gk,e0}$ from the Laplace transform of Eqs. (4.278)–(4.281) one gets the equations for the Laplace transforms of the matrix elements of the density matrix,

$$\begin{aligned} z\tilde{\rho}_{e0,e0}(z) - 1 &= - \sum_k |g(k)|^2 \left(\frac{1}{z + i(\omega_k - \omega_{eg}) + \frac{1}{\tau_M}} + \frac{1}{z + i(\omega_{eg} - \omega_k) + \frac{1}{\tau_M}} \right) \tilde{\rho}_{e0,e0}(z) \\ &+ \sum_{k,k'} \left(\frac{1}{z + i(\omega_k - \omega_{eg}) + \frac{1}{\tau_M}} + \frac{1}{z + i(\omega_{eg} - \omega_{k'}) + \frac{1}{\tau_M}} \right) \\ &\times g(k)g(k')^* \tilde{\rho}_{gk,gk'}(z), \end{aligned} \quad (4.282)$$

$$\begin{aligned} (z + i\omega_{kk'})\tilde{\rho}_{gk,gk'}(z) &= - \sum_{k''} \left(\frac{g(k')g(k'')^*}{z + i(\omega_k - \omega_{eg}) + \frac{1}{\tau_M}} \tilde{\rho}_{gk,gk''}(z) + \frac{g(k'')g(k)^*}{z + i(\omega_{eg} - \omega_{k'}) + \frac{1}{\tau_M}} \tilde{\rho}_{gk'',gk'}(z) \right) \\ &+ g(k')g(k)^* \left(\frac{1}{z + i(\omega_k - \omega_{eg}) + \frac{1}{\tau_M}} + \frac{1}{z + i(\omega_{eg} - \omega_{k'}) + \frac{1}{\tau_M}} \right) \tilde{\rho}_{e0,e0}(z). \end{aligned} \quad (4.283)$$

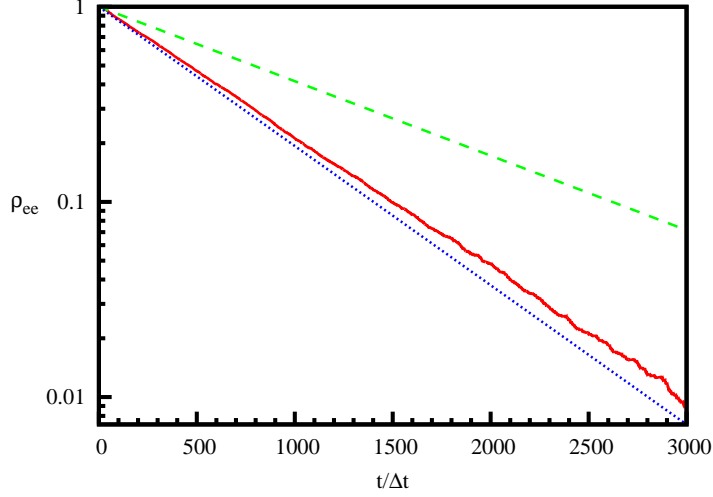


Figure 4.15: Time dependence of the occupation of the exited level of the decaying system when the interaction with the reservoir modes is described by Eq. (4.262). Solid line shows results of the numerical calculation, dashed line shows exponential decay of the measurement-free system with the decay rate given by Eq. (4.263). The dotted line shows approximation according to Eq. (4.285). In the calculations we used $a = 2$, while other parameters are the same as in Fig. 4.12.

On the r.h.s. of Eq. (4.283) we will neglect the small terms not containing $\tilde{\rho}_{e0,e0}(z)$. Expressing $\tilde{\rho}_{gk,gk'}(z)$ via $\tilde{\rho}_{e0,e0}(z)$ from Eq. (4.283), substituting into Eq. (4.282) and replacing the sum over k by an integral we obtain

$$\begin{aligned} \frac{1}{\tilde{\rho}_{e0,e0}(z)} &= z + \int d\omega G(\omega) \left(\frac{1}{z + i(\omega - \omega_{eg}) + \frac{1}{\tau_M}} + \frac{1}{z + i(\omega_{eg} - \omega) + \frac{1}{\tau_M}} \right) \\ &\quad - \int d\omega \int d\omega' G(\omega) G(\omega') \frac{1}{z + i(\omega - \omega')} \\ &\quad \times \left(\frac{1}{z + i(\omega - \omega_{eg}) + \frac{1}{\tau_M}} + \frac{1}{z + i(\omega_{eg} - \omega') + \frac{1}{\tau_M}} \right)^2. \end{aligned} \quad (4.284)$$

The value of z at which the r.h.s of Eq. (4.284) is equal to zero gives the decay rate. Using the expression (4.277) for $G(\omega)$ and keeping only the first-order terms of the expansion into series of the powers of Λ^{-1} we get the measurement-modified decay rate

$$\Gamma_{e \rightarrow g} = \Gamma_{e \rightarrow g}^{(0)} \left(1 - \frac{\Gamma_{e \rightarrow g}^{(0)}}{\pi \Lambda} (5a^2 - 1) \right) + \Gamma_{e \rightarrow g}^{(0)} \frac{2}{\pi} \frac{(a^2 - 1)}{\Lambda \tau_M}. \quad (4.285)$$

Equation (4.285) is valid only for sufficiently large duration of the measurement τ_M , since expansion into series requires that $\Lambda \tau_M \gg 1$. From Eq. (4.285) one can see that in order to obtain the quantum anti-Zeno effect the parameter a should be greater than 1. When the parameter a is less than 1 we get the Zeno effect, and when $a = 1$ the decay rate coincides with the decay rate of the free system.

The probability that the measured system stays in the excited state, obtained from numerical simulation is presented in Fig. 4.15. Figure clearly demonstrates the quantum anti-Zeno effect, the decay rate of the measured system is bigger than that of the measurement-free system.

4.8 Summary

In this work we investigate the quantum Zeno effect using different models of the measurement. We take into account the finite duration and the finite accuracy of the measurement. At first we consider a simple model of the measurement without taking into account the interaction of the detector with the environment. Using this model of the measurement the general equation for the probability of the jump during the measurement is derived (4.36). The behavior of the system under the repeated measurements depends on the strength of measurement and on the properties of the system.

When the the strength of the interaction with the measuring device is sufficiently large, the frequent measurements of the system with discrete spectrum slow down the evolution. However, the evolution cannot be fully stopped. Under the repeated measurements the occupation of the energy levels changes exponentially with time, approaching the limit of the equal occupation of the levels. The jump probability is inversely proportional to the strength of the interaction with the measuring device.

In the case of a continuous spectrum the measurements can cause inhibition or acceleration of the evolution. Our model of the continuous measurement gives the same result as the approach based on the projection postulate [26]. The decay rate is equal to the convolution of the reservoir coupling spectrum with the measurement-modified shape of the spectral line. The width of the spectral line is proportional to the strength of the interaction with the measuring device. When this width is much greater than the width of the reservoir, the quantum Zeno effect takes place. Under these conditions the decay rate is inversely proportional to the strength of the interaction with the measuring device. In a number of decaying systems, however, the reservoir spectrum $G(\omega)$ grows with frequency almost up to the relativistic cut-off and the strength of the interaction required for the appearance of the quantum Zeno effect is so high that the initial system is significantly modified. When the spectral line is not very broad, the decay rate may be increased by the measurements more often than it may be decreased and the quantum anti-Zeno effect can be obtained.

The quantum Zeno effect is often analysed using the succession of the instantaneous measurements with free evolution of the measured system between the measurements. We analyze the measurements with finite duration, instead. We apply the simple model of the measurement, developed in Ref. [47]. The equations for the jump probability (4.86)-(4.89) are obtained. Applying the equations to the measured two-level system we obtain a simple expression for the probability of the jump from one level to the other (4.104). The influence of the finite duration of the measurement is expressed as the small correction.

Furthher we analyze the quantum Zeno and quantum anti-Zeno effects without using any particular model of the measurement. The general expression (4.126) for the jump probability during the measurement is derived. The main assumptions, used in the derivation of Eq. (4.126), are the assumptions that the quantum measurement is non-demolition

measurement (Eq. (4.107)) and the Markovian approximation for the quantum dynamics is valid (Eq. (4.111)). We have shown that Eq. (4.126) is also suitable for the description of the pulsed measurements, when there are intervals of the measurement-free evolution between the successive measurements (Eqs. (4.132)–(4.135)). When the operator \hat{V} inducing the jumps from one state to another does not depend on time Eq. (4.141), which is of the form obtained by Kofman and Kurizki [26], is derived as a special case.

We apply the equations derived in Sec. 4.5 using a concrete irreversible model of the measurement. The detector is modeled as a harmonic oscillator, initially being at the thermal equilibrium. The interaction of the detector with the system is modeled similarly as in Ref. [47]. The Lindblad-type master equation for the detectors density matrix is solved analytically. An equation for the probability of the jump between measured system's states during the measurement, similar to that of Refs. [26, 47, 60], is obtained (4.185). From the used model it follows that the increase of the detector's temperature leads to the enhancement of the quantum Zeno or quantum anti-Zeno effects.

Another model of the detector is considered in Sec. 4.7. The detector is a two level system interacting with the environment. The influence of the environment is taken into account using quantum trajectory method. The quantum trajectories produced by stochastic simulations show the probabilistic behavior exhibiting the collapse of the wavepacket in the measured system, although the quantum jumps were performed only in the detector. Both quantum Zeno and anti-Zeno effects were demonstrated for the measured two level system and for the decaying system. The results of the numerical calculations are compared with the analytical expressions for the decay rate of the measured system. It is found that the general expression (4.242), obtained in Ref. [77], gives good agreement with the numerical data for the measured two level system and for the decaying one showing the quantum Zeno effect. Nevertheless, when the interaction of the measured system with the reservoir is strongly mode-dependent, this expression does not give the correct decay rate. The decay rate in this case was estimated including additional terms describing decay of non-diagonal elements into the equation for the density matrix of the measured system and a good agreement with the numerical calculations is found. A good agreement of the numerical results with the analytical estimates of the decay rates of the measured system shows that the particular model of the detector is not important, since the decay rates mostly depend only on one parameter, i.e., the duration of the measurement (4.206).

5 Weak measurements and time problem in quantum mechanics

5.1 Introduction

Time plays a special role in quantum mechanics. Unlike other observables, time remains a classical variable. It cannot be simply quantized because, as it is well known, the self-adjoint operator of time does not exist for bounded Hamiltonians. The problems with the time also rise from the fact that in quantum mechanics many quantities cannot have definite values simultaneously. The absence of the time operator makes this problem even more complicated. However, in practice the time often is important for an experimenter. If quantum mechanics can correctly describe the outcomes of the experiments, it must also give the method for the calculation of the time the particle spends in some region.

The most-known problem of time in quantum mechanics is the so-called "tunneling time problem". Tunneling phenomena are inherent in numerous quantum systems, ranging from atom and condensed matter to quantum fields. Therefore, the questions about the tunneling mechanisms are important. There have been many attempts to define a physical time for tunneling processes, since this question has been raised by MacColl [80] in 1932. This question is still the subject of much controversy, since numerous theories contradict each other in their predictions for "the tunneling time". Some of these theories predict that the tunneling process is faster than light, whereas the others state that it should be subluminal. This subject has been covered in a number of reviews (Hauge and Støvneng [81], 1989; Olkholovsky and Recami [82], 1992; Landauer and Martin [83], 1994 and Chiao and Steinberg [84], 1997). The fact that there is a time related to the tunneling process has been observed experimentally [85–93]. However, the results of the experiments are ambiguous.

Many problems with time in quantum mechanics arise from the noncommutativity of the operators in quantum mechanics. The noncommutativity of the operators in quantum mechanics can be circumvented by using the concept of weak measurements. The concept of weak measurement was proposed by Aharonov, Albert and Vaidman [94–99]. Such an approach has several advantages. It gives, in principle, the procedure for measuring the physical quantity. Since in the classical mechanics all quantities can have definite values simultaneously, weak measurements give the correct classical limit. The concept of weak measurements has been already applied to the time problem in quantum mechanics [53, 100, 101].

The time in classical mechanics describes not a single state of the system but the process of the evolution. This property is an essential concept of the time. We speak about the time belonging to a certain evolution of the system. If the measurement of the time disturbs the evolution we cannot attribute this measured duration to the undisturbed

evolution. Therefore, we should require that the measurement of the time does not disturb the motion of the system. This means that the interaction of the system with the measuring device must be weak. In quantum mechanics this means that we cannot use the strong measurements described by the von-Neumann's projection postulate. We have to use the weak measurements of Aharonov, Albert and Vaidman [94–99], instead.

This Chapter is organized as follows: In Sec. 5.2 we present the model of the weak measurements. Sec. 5.3 presents the time on condition that the system is in the given final state. In Sec. 5.4 our formalism is applied for the tunneling time problem. In Sec. 5.5 the weak measurement of the quantum arrival time distribution is presented. Section 5.6 summarizes our findings.

5.2 The concept of weak measurements

In this section we present the concept of weak measurement, proposed by Aharonov, Albert and Vaidman [94–99]. We measure quantity represented by the operator \hat{A}

We have the detector in the initial state $|\Phi\rangle$. In order the weak measurements can provide the meaningful information, the measurements must be performed on an ensemble of identical systems. Each system with its own detector is prepared in the same initial state. The readings of the detectors are collected and averaged.

Our model consists of the system \mathbf{S} under consideration and of the detector \mathbf{D} . The Hamiltonian is

$$\hat{H} = \hat{H}_S + \hat{H}_D + \hat{H}_I \quad (5.1)$$

where \hat{H}_S and \hat{H}_D are the Hamiltonians of the system and of the detector, respectively. We take the operator describing the interaction between the particle and the detector of the form [47, 53, 58, 59, 77, 100]

$$\hat{H}_I = \lambda \hat{q} \hat{A}, \quad (5.2)$$

λ characterizes the strength of the interaction with the detector. A very small parameter λ ensures the undisturbance of the system's evolution. The measurement time is τ . In this section we assume that the interaction strength λ and the time τ are small. The operator \hat{q} acts in the Hilbert space of the detector. We require a continuous spectrum of the operator \hat{q} . For simplicity, we can consider this operator as the coordinate of the detector. The momentum, conjugate to q , is p_q .

The interaction operator (5.2) only slightly differs from the one used by Aharonov, Albert and Vaidman [95]. The similar interaction operator has been considered by von Neumann [102] and has been widely used in the strong measurement models (e.g., [43–46, 58, 103] and many others).

Hamiltonian (5.2) represents the constant force acting on the detector. This force results in the change of the detector's momentum. From the classical point of view, the change of the momentum is proportional to the force acting on the detector. Since interaction strength λ and the duration of the measurement τ are small, the average $\langle \hat{A} \rangle$ does not change significantly during the measurement. The action of the Hamiltonian (5.2) results in the small change of the mean detector's momentum $\langle \hat{p}_q \rangle - \langle \hat{p}_q \rangle_0 = -\lambda \tau \langle \hat{A} \rangle$, where $\langle \hat{p}_q \rangle_0 = \langle \Phi(0) | \hat{p}_q | \Phi(0) \rangle$ is the mean momentum of the detector at the beginning of the measurement and $\langle \hat{p}_q \rangle = \langle \Phi(\tau) | \hat{p}_q | \Phi(\tau) \rangle$ is the mean momentum of the detector after

the measurement. Therefore, in analogy to Ref. [95], we define the “weak value” of the average $\langle \hat{A} \rangle$,

$$\langle \hat{A} \rangle = \frac{\langle \hat{p}_q \rangle_0 - \langle \hat{p}_q \rangle}{\lambda \tau}. \quad (5.3)$$

In the time moment $t = 0$ the density matrix of the whole system is $\hat{\rho}(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_D(0)$, where $\hat{\rho}_S(0)$ is the density matrix of the system and $\hat{\rho}_D(0) = |\Phi\rangle\langle\Phi|$ is the density matrix of the detector. After the interaction the density matrix of the detector is $\hat{\rho}_D(t) = \text{Tr}_S \left\{ \hat{U}(t) (\hat{\rho}_S(0) \otimes |\Phi\rangle\langle\Phi|) \hat{U}^\dagger(t) \right\}$ where $\hat{U}(t)$ is the evolution operator. Later on, for simplicity we will neglect the Hamiltonian of the detector. Then, the evolution operator in the first-order approximation is [100]

$$\hat{U}(t) \approx \hat{U}_S(t) \left(1 + \frac{\lambda \hat{q}}{i\hbar} \int_0^t \tilde{A}(t_1) dt_1 \right) \quad (5.4)$$

where $\hat{U}_S(t)$ is the evolution operator of the unperturbed system and $\tilde{A}(t) = \hat{U}_S^\dagger(t) \hat{A} \hat{U}_S(t)$. From Eq. (5.3) we obtain that the weak value coincides with the usual average $\langle \hat{A} \rangle = \text{Tr} \{ \hat{A} \hat{\rho}_S(0) \}$.

The influence of the weak measurement on the evolution of the measured system can be made arbitrary small using small parameter λ . Therefore, after the interaction of the measured system with the detector we can try to measure second observable \hat{B} using usual, strong measurement. As far as our model gives the correct result for the value of A averaged over the entire ensemble of the systems, we can try to take the average only over the subensemble of the systems with the given value of the quantity B . We measure the momenta p_q of each measuring device after the interaction with the system. Subsequently we perform the final, postselection measurement of B on the systems of our ensemble. Then we collect the outcomes p_q only of the systems which have given value of B .

The joint probability that the system has the given value of B and the detector has the momentum $p_q(t)$ at the time moment t is $W(B, p_q; t) = \text{Tr} \{ |B\rangle\langle B| p_q \langle p_q | \hat{\rho}(t) \}$, where $|p_q\rangle$ is the eigenfunction of the momentum operator \hat{p}_q . In quantum mechanics the probability that two quantities simultaneously have definite values does not always exist. If the joint probability does not exist then the concept of the conditional probability is meaningless. However, in our case operators \hat{B} and $|p_q\rangle\langle p_q|$ act in different spaces and commute, therefore, the probability $W(B, p_q; t)$ exists.

Let us define the conditional probability, i.e., the probability that the momentum of the detector is p_q provided that the system has the given value of B . This probability is given according to Bayes theorem as

$$W(p_q; t|B) = \frac{W(B, p_q; t)}{W(B; t)} \quad (5.5)$$

where $W(B; t) = \text{Tr} \{ |B\rangle\langle B| \hat{\rho}(t) \}$ is the probability that the system has the given value of B . The average momentum of the detector on condition that the system has the given value of B is

$$\langle p_q(t) \rangle = \int p_q W(p_q; t|B) dp_q. \quad (5.6)$$

From Eqs. (5.3) and (5.6), in the first-order approximation we obtain the mean value of A on condition that the system has the given value of B (see for analogy Ref. [100])

$$\begin{aligned} \langle A \rangle_B &= \frac{1}{2\langle B|\hat{\rho}_S|B\rangle} \left\langle |B\rangle\langle B|\hat{A} + \hat{A}|B\rangle\langle B| \right\rangle \\ &+ \frac{1}{i\hbar\langle B|\hat{\rho}_S|B\rangle} (\langle q \rangle\langle p_q \rangle - \text{Re}\langle \hat{q}\hat{p}_q \rangle) \left\langle [|B\rangle\langle B|, \hat{A}] \right\rangle. \end{aligned} \quad (5.7)$$

If the commutator $[|B\rangle\langle B|, \hat{A}]$ in Eq. (5.7) is not zero then, even in the limit of the very weak measurement, the measured value depends on the particular detector. This fact means that in such a case we cannot obtain the *definite* value. Moreover, the coefficient $(\langle q \rangle\langle p_q \rangle - \text{Re}\langle \hat{q}\hat{p}_q \rangle)$ may be zero for the specific initial state of the detector, e.g., for the Gaussian distribution of the coordinate q and momentum p_q .

5.3 The time on condition that the system is in the given final state

The most-known problem of time in quantum mechanics is the so-called "tunneling time problem". We can raise another, more general, question about the time. Let us consider a system which evolves with time. Let χ is one of the observables of the system. During the evolution the value of χ changes. We are considering a subset Γ of possible values of χ . The question is *how much time the values of χ belong to this subset?*

There is another version of the question. If we know the final state of the system, we may ask how much time the values of χ belong to the subset under consideration when the system evolves from the initial to the definite final state. The question about the tunneling time belongs to such class of the problems. Really, in the tunneling time problem we ask about the duration the particle spends in a specified region of the space and we know that the particle has tunneled, i.e., it is on the other side of the barrier. We can expect that such a question may not always be answered. Here our goal is to obtain the conditions under which it is possible to answer such a question.

One of the possibilities to solve the problem of time is to answer what exactly the word "time" means. The meaning of every quantity is determined by the procedure of measurement. Therefore, we have to construct a scheme of an experiment (this can be a *gedanken* experiment) to measure the quantity with the properties corresponding to the classical time.

5.3.1 The model of the time measurement

We consider a system evolving with time. One of the quantities describing the system is χ . Operator $\hat{\chi}$ corresponds to this quantity. For simplicity we assume that the operator $\hat{\chi}$ has a continuous spectrum. The case with discrete spectrum will be considered later.

The measuring device interacts with the system only if χ is near some point χ_D , depending only on the detector. If we want to measure the time the system is in a large region of χ , we have to use many detectors. In the case of tunneling a similar model had been introduced by Steinberg [104] and developed in our paper [53]. The strong limit of

such a model for analysis of the measurement effect for the quantum jumps has been used in Ref. [47].

We use the weak measurement, described in Sec. 5.2. As the operator \hat{A} we take the operator

$$\hat{D}(\chi_D) = |\chi_D\rangle\langle\chi_D| = \delta(\hat{\chi} - \chi_D). \quad (5.8)$$

After time t the readings of the detectors are collected and averaged.

Hamiltonian (5.2) with \hat{D} given by (5.8) represents the constant force acting on the detector \mathbf{D} when the quantity χ is very close to the value χ_D . This force induces the change of the detector's momentum. From the classical point of view, the change of the momentum is proportional to the time the particle spends in the region around χ_D and the coefficient of proportionality equals the force acting on the detector. We assume that the change of the mean momentum of the detector is proportional to the time the constant force acts on the detector and that the time the particle spends in the detector's region is the same as the time the force acts on the detector.

We can replace the δ function by the narrow rectangle of height $1/L$ and of width L in the χ space. From Eq. (5.2) it follows that the force acting on the detector when the particle is in the region around χ_D is $F = -\lambda/L$. The time the particle spends until time moment t in the unit-length region is

$$\tau(t) = -\frac{1}{\lambda} (\langle p_q(t) \rangle - \langle p_q \rangle) \quad (5.9)$$

where $\langle p_q \rangle$ and $\langle p_q(t) \rangle$ are the mean initial momentum and momentum after time t , respectively. If we want to find the time the system spends in the region of the finite width, we have to add many times (5.9).

When the operator $\hat{\chi}$ has a discrete spectrum, we may ask how long the quantity χ has the value χ_D . To answer this question the detector must interact with the system only when $\chi = \chi_D$. In such a case the operator $\hat{D}(\chi_D)$ takes the form

$$\hat{D}(\chi_D) = |\chi_D\rangle\langle\chi_D|. \quad (5.10)$$

The force, acting on the detector in this case equals to $-\lambda$. The time the quantity χ has the value χ_D is given by Eq. (5.9), too. Further formulae do not depend on the spectrum of the operator $\hat{\chi}$.

5.3.2 The dwell time

For shortening of the notation we introduce the operator

$$\hat{F}(\chi_D, t) = \int_0^t \tilde{D}(\chi_D, t_1) dt_1, \quad (5.11)$$

where

$$\tilde{D}(\chi_D, t) = \hat{U}_S^\dagger(t) \hat{D}(\chi_D) \hat{U}_S(t). \quad (5.12)$$

From the density matrix of the detector after the measurement in the first-order approximation we find that the average change of the momentum of the detector during the time t is $-\lambda \langle \hat{F}(\chi_D, t) \rangle$. From Eq. (5.9) we obtain the dwell time until time moment t ,

$$\tau(\chi, t) = \left\langle \hat{F}(\chi, t) \right\rangle. \quad (5.13)$$

The time spent in the region Γ is

$$t(\Gamma; t) = \int_{\Gamma} \tau(\chi, t) d\chi = \int_0^t dt' \int_{\Gamma} d\chi P(\chi, t'), \quad (5.14)$$

where $P(\chi, t') = \langle \tilde{D}(\chi, t') \rangle$ is the probability for the system to have the value χ at time moment t' .

In the case when χ is the coordinate of the particle Eq. (5.14) yields the well-known expression for the dwell time [53, 82]. This time is the average over the entire ensemble of the systems, regardless of their final states.

5.3.3 The time on condition that the system is in the given final state

Further we will consider the case when the final state of the system is known. We ask *how much time the values of χ belong to the subset under consideration, Γ , on condition that the system evolves to the definite final state f* . More generally, we might know that the final state of the system belongs to the certain subspace \mathcal{H}_f of system's Hilbert space.

The projection operator which projects the vectors from the Hilbert space of the system into the subspace \mathcal{H}_f of the final states is \hat{P}_f . As far as our model gives the correct result for the time averaged over the entire ensemble of the systems, we can try to take the average only over the subensemble of the systems with the given final states. We measure the momenta p_q of each measuring device after the interaction with the system. Subsequently we perform the final, postselection measurement on the systems of our ensemble. Then we collect the outcomes p_q only of the systems the final state of which turns out to belong to the subspace \mathcal{H}_f .

Using Eq. (5.7) from Sec. 5.2 we obtain the duration on condition that the final state of the system belongs to the subspace \mathcal{H}_f [100]

$$\begin{aligned} \tau_f(\chi, t) &= \frac{1}{2\langle \tilde{P}_f(t) \rangle} \left\langle \tilde{P}_f(t) \hat{F}(\chi, t) + \hat{F}(\chi, t) \tilde{P}_f(t) \right\rangle \\ &+ \frac{1}{i\hbar\langle \tilde{P}_f(t) \rangle} (\langle q \rangle \langle p_q \rangle - \text{Re}\langle \hat{q}\hat{p}_q \rangle) \left\langle \left[\tilde{P}_f(t), \hat{F}(\chi, t) \right] \right\rangle. \end{aligned} \quad (5.15)$$

Eq. (5.15) consists of two terms and we can introduce two expressions with the dimension of time

$$\tau_f^{(1)}(\chi, t) = \frac{1}{2\langle \tilde{P}_f(t) \rangle} \left\langle \tilde{P}_f(t) \hat{F}(\chi, t) + \hat{F}(\chi, t) \tilde{P}_f(t) \right\rangle, \quad (5.16)$$

$$\tau_f^{(2)}(\chi, t) = \frac{1}{2i\langle \tilde{P}_f(t) \rangle} \left\langle \left[\tilde{P}_f(t), \hat{F}(\chi, t) \right] \right\rangle. \quad (5.17)$$

Then the time the system spends in the subset Γ on condition that the final state of the system belongs to the subspace \mathcal{H}_f can be rewritten in the form

$$\tau_f(\chi, t) = \tau_f^{(1)}(\chi, t) + \frac{2}{\hbar} (\langle q \rangle \langle p_q \rangle - \text{Re}\langle \hat{q}\hat{p}_q \rangle) \tau_f^{(2)}(\chi, t). \quad (5.18)$$

5.3 The time on condition that the system is in the given final state

The quantities $\tau_f^{(1)}(\chi, t)$ and $\tau_f^{(2)}(\chi, t)$ are related to the real and imaginary parts of the complex time, introduced by D. Sokolovski *et. al* [105]. In our model the quantity $\tau_f(\chi, t)$ is real, contrary to the complex-time approach. The components of time $\tau_f^{(1)}$ and $\tau_f^{(2)}$ are real, too. Therefore, this time can be interpreted as the duration of an event.

If the commutator $[\tilde{P}_f(t), \hat{F}(\chi, t)]$ in Eq. (5.15) is not zero then, even in the limit of the very weak measurement, the measured value depends on the particular detector. This fact means that in such a case we cannot obtain the *definite* value for the conditional time. Moreover, the coefficient $(\langle q \rangle \langle p_q \rangle - \text{Re} \langle \hat{q} \hat{p}_q \rangle)$ may be zero for the specific initial state of the detector, e.g., for the Gaussian distribution of the coordinate q and momentum p_q .

The conditions of the possibility to determine the time uniquely in a case when the final state of the system is known takes the form

$$[\tilde{P}_f(t), \hat{F}(\chi, t)] = 0. \quad (5.19)$$

This result can be understood basing on general principles of quantum mechanics, too. We ask *how much time the values of χ belong to the certain subset when the system evolves to the given final state*. We know with certainty the final state of the system. In addition, we want to have some information about the values of the quantity χ . However, if we know the final state with certainty, we may not know the values of χ in the past and, vice versa, if we know something about χ , we may not definitely determine the final state. Therefore, in such a case the question about the time when the system evolves to the given final state cannot be answered definitely and the conditional time has no reasonable meaning.

The quantity $\tau_f(t)$ according to Eqs. (5.15) and (5.16) has many properties of the classical time. So, if the final states $\{f\}$ constitute the full set, then the corresponding projection operators obey the equality of completeness $\sum_f \hat{P}_f = 1$. Then from Eq. (5.15) we obtain the expression

$$\sum_f \langle \tilde{P}_f(t) \rangle \tau_f(\chi, t) = \tau(\chi, t). \quad (5.20)$$

The quantity $\langle \tilde{P}_f(t) \rangle$ is the probability that the system at the time t is in the state f . Eq. (5.20) shows that the full duration equals to the average over all possible final states, as it is a case in the classical physics. From Eq. (5.20) and Eqs. (5.16), (5.17) it follows

$$\sum_f \langle \tilde{P}_f(t) \rangle \tau_f^{(1)}(\chi, t) = \tau(\chi, t), \quad (5.21)$$

$$\sum_f \langle \tilde{P}_f(t) \rangle \tau_f^{(2)}(\chi, t) = 0. \quad (5.22)$$

We suppose that quantities $\tau_f^{(1)}(\chi, t)$ and $\tau_f^{(2)}(\chi, t)$ can be useful even in the case when the time has no definite value, since in the tunneling time problem the quantities (5.16) and (5.17) correspond to the real and imaginary parts of the complex time, respectively [53].

The eigenfunctions of the operator $\hat{\chi}$ constitute the full set $\int |\chi\rangle \langle \chi| d\chi = 1$ where the integral must be replaced by the sum for the discrete spectrum of the operator $\hat{\chi}$. From Eqs. (5.8), (5.11), (5.15) we obtain the equality

$$\int \tau_f(\chi, t) d\chi = t. \quad (5.23)$$

Eq. (5.23) shows that the time during which the quantity χ has any value equals to t , as it is in the classical physics.

5.3.4 Example: two-level system

The obtained formalism can be applied to the tunneling time problem [53]. In this section, however, we will consider a simpler system than the tunneling particle, i.e., a two-level system. The system is forced by the perturbation V which causes the jumps from one state to another. We will determine the time the system is in the given state.

The Hamiltonian of this system is

$$\hat{H} = \hat{H}_0 + \hat{V} \quad (5.24)$$

where $\hat{H}_0 = \hbar\omega\hat{\sigma}_3/2$ is the Hamiltonian of the unperturbed system and $\hat{V} = v\hat{\sigma}_+ + v^*\hat{\sigma}_-$ is the perturbation. Here $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices and $\sigma_{\pm} = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$. The Hamiltonian \hat{H}_0 has two eigenfunctions $|0\rangle$ and $|1\rangle$ with the eigenvalues $-\hbar\omega/2$ and $\hbar\omega/2$, respectively. The initial state of the system is $|0\rangle$.

From Eq. (5.13) we obtain the times the system spends in the energy levels 0 and 1, respectively,

$$\tau(0, t) = \frac{1}{2} \left(1 + \frac{\omega^2}{\Omega^2} \right) t + \frac{1}{2\Omega} \sin(\Omega t) \left(1 - \frac{\omega^2}{\Omega^2} \right), \quad (5.25)$$

$$\tau(1, t) = \frac{1}{2} \left(1 - \frac{\omega^2}{\Omega^2} \right) t - \frac{1}{2\Omega} \sin(\Omega t) \left(1 - \frac{\omega^2}{\Omega^2} \right) \quad (5.26)$$

where $\Omega = \sqrt{\omega^2 + 4(|v|/\hbar)^2}$. From Eqs. (5.16) and (5.17) we can obtain the conditional time. The components $\tau^{(1)}$ (5.16) and $\tau^{(2)}$ (5.17) of the time the system spends in the level 0 on condition that the final state is $|1\rangle$ are

$$\tau_1^{(1)}(0, t) = \frac{t}{2}, \quad (5.27)$$

$$\tau_1^{(2)}(0, t) = \frac{\omega}{2\Omega} \left(1 - t \cot \left(\frac{\Omega}{2} t \right) \right). \quad (5.28)$$

When $\Omega t = 2\pi n$, $n \in \mathbb{Z}$, the quantity $\tau_1^{(2)}(0, t)$ tends to infinity. This is because at these time moments the system is in state $|1\rangle$ with the probability 0 and we cannot consider the interaction with the detector as very weak.

The components of the time (5.16) and (5.17) the system spends in level 0 on condition that the final state is $|0\rangle$ are

$$\tau_0^{(1)}(0, t) = \frac{\left(1 + 3\frac{\omega^2}{\Omega^2} \right) t + \left(1 - \frac{\omega^2}{\Omega^2} \right) \left(\frac{2}{\Omega} \sin(\Omega t) + t \cos(\Omega t) \right)}{2 \left(\left(1 + \frac{\omega^2}{\Omega^2} \right) + \left(1 - \frac{\omega^2}{\Omega^2} \right) \cos(\Omega t) \right)}, \quad (5.29)$$

$$\tau_0^{(2)}(0, t) = \frac{\frac{\omega}{\Omega} \left(1 - \frac{\omega^2}{\Omega^2} \right) \sin \left(\frac{\Omega}{2} t \right) \left(t \cos \left(\frac{\Omega}{2} t \right) - \frac{2}{\Omega} \sin \left(\frac{\Omega}{2} t \right) \right)}{2 \left(\left(1 + \frac{\omega^2}{\Omega^2} \right) + \left(1 - \frac{\omega^2}{\Omega^2} \right) \cos(\Omega t) \right)}. \quad (5.30)$$

5.3 The time on condition that the system is in the given final state

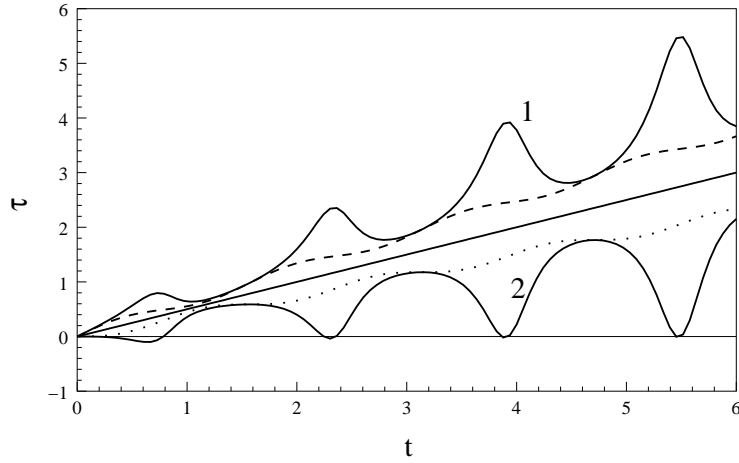


Figure 5.1: The times the system spends in the energy levels 0, $\tau(0,t)$ (dashed line) and level 1, $\tau(1,t)$ (dotted line), according to Eqs. (5.25) and (5.26), respectively. The quantity $\tau_1^{(1)}(0,t)$, Eq. (5.27), is shown as solid straight line. The quantities $\tau_0^{(1)}(0,t)$ (1) and $\tau_0^{(1)}(1,t)$ (2) are calculated according to Eqs. (5.29) and (5.31), respectively. The parameters are $\omega = 2$ and $\Omega = 4$.

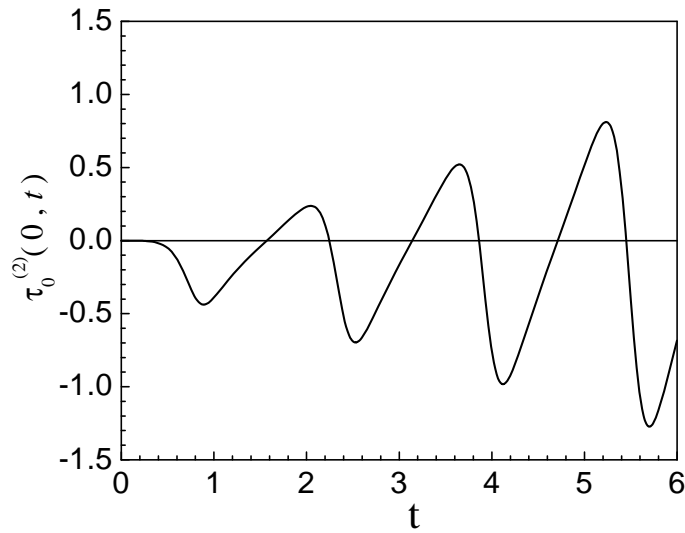


Figure 5.2: The quantity $\tau_0^{(2)}(0,t)$, Eq. (5.30). The parameters are the same as in Fig. 5.1.

The time the system spends in level 1 on condition that the final state is $|0\rangle$ may be expressed as

$$\tau_0^{(1)}(1, t) = \frac{\left(1 - \frac{\omega^2}{\Omega^2}\right) \left(t + t \cos(\Omega t) - \frac{2}{\Omega} \sin(\Omega t)\right)}{2 \left(\left(1 + \frac{\omega^2}{\Omega^2}\right) + \left(1 - \frac{\omega^2}{\Omega^2}\right) \cos(\Omega t)\right)}. \quad (5.31)$$

The quantities $\tau(0, t)$, $\tau(1, t)$, $\tau_1^{(1)}(0, t)$, $\tau_0^{(1)}(0, t)$ and $\tau_0^{(1)}(1, t)$ are shown in Fig. 5.1. The quantity $\tau_0^{(2)}(0, t)$ is shown in Fig. 5.2. Note that the duration with the given final state is not necessarily monotonic as it is with the full duration because this final state at different time moments can be reached by different ways. We can interpret the quantity $\tau_0^{(1)}(0, t)$ as the time the system spends in the level 0 on condition that the final state is $|0\rangle$, but at certain time moments this quantity is greater than t . The quantity $\tau_0^{(1)}(1, t)$ becomes negative at certain time moments. This is the consequence of the fact that for the system under consideration the condition (5.19) is not fulfilled. The peculiarities of the behavior of the conditional times show that it is impossible to decompose the unconditional time into two components having all classical properties of the time.

5.4 Tunneling time

The most-known problem of time in quantum mechanics is the so-called "tunneling time problem". This problem is still the subject of much controversy, since numerous theories contradict each other in their predictions for "the tunneling time". Many of the theoretical approaches can be divided into three categories. First, one can study evolution of the wave packets through the barrier and get the phase time. However, the correctness of the definition of this time is highly questionable [106]. Another approach is based on the determination of a set of dynamic paths, i.e., the calculation of the time the different paths spend in the barrier and averaging over the set of the paths. The paths can be found from the Feynman path integral formalism [105], from the Bohm approach [107–110], or from the Wigner distribution [111]. The third class uses a physical clock which can be used for determination of the time elapsed during the tunneling (Büttiker and Landauer used an oscillatory barrier [106], Baz' suggested the Larmor time [112]).

The problems rise also from the fact that the arrival time of a particle to the definite spatial point is a classical concept. Its quantum counterpart is problematic even for the free particle case. In classical mechanics, for the determination of the time the particle spends moving along a certain trajectory, we have to measure the position of the particle at two different moments of time. In quantum mechanics this procedure does not work. From Heisenberg's uncertainty principle it follows that we cannot measure the position of a particle without alteration of its momentum. To determine exactly the arrival time of a particle, one has to measure the position of the particle with great precision. Because of the measurement, the momentum of the particle will have a big uncertainty and the second measurement will be indefinite. If we want to ask about the time in quantum mechanics, we need to define the procedure of measurement. We can measure the position of the particle only with a finite precision and get a distribution of the possible positions. Applying such a measurement, we can expect to obtain not a single value of the traversal time but a distribution of times.

The question of *how much time does the tunneling particle spend in the barrier region* is not precise. There are two different but related questions connected with the tunneling time problem [113]:

- (i). How much time does the tunneling particle spend under the barrier?
- (ii). At what time does the particle arrive at the point behind the barrier?

There have been many attempts to answer these questions. However, there are several papers showing that according to quantum mechanics the question (i) makes no sense [113–116]. Our goal is to investigate the possibility to determine the tunneling time using weak measurements.

5.4.1 Determination of the tunneling time

To answer the question of *how much time does the tunneling particle spends under the barrier*, we need a criterion of the tunneling. We accept the following criterion: the particle had tunneled in the case when it was in front of the barrier at first and later it was found behind the barrier. We require that the mean energy of the particle and the energy uncertainty must be less than the height of the barrier. Following this criterion, we introduce an operator corresponding to the “tunneling-flag” observable

$$\hat{f}_T(X) = \Theta(\hat{x} - X) \quad (5.32)$$

where Θ is the Heaviside unit step function and X is a point behind the barrier. This operator projects the wave function onto the subspace of functions localized behind the barrier. The operator has two eigenvalues: 0 and 1. The eigenvalue 0 corresponds to the fact that the particle has not tunneled, while the eigenvalue 1 corresponds to the tunneled particle.

We will work with the Heisenberg representation. In this representation, the tunneling flag operator is

$$\tilde{f}_T(t, X) = \exp\left(\frac{i}{\hbar}\hat{H}t\right) \hat{f}_T(X) \exp\left(-\frac{i}{\hbar}\hat{H}t\right). \quad (5.33)$$

To take into account all the tunneled particles, the limit $t \rightarrow +\infty$ must be taken. So, the “tunneling-flag” observable in the Heisenberg picture is represented by the operator $\tilde{f}_T(\infty, X) = \lim_{t \rightarrow +\infty} \tilde{f}_T(t, X)$. We can obtain an explicit expression for this operator.

The operator $\tilde{f}_T(t, X)$ obeys the equation

$$i\hbar \frac{\partial}{\partial t} \tilde{f}_T(t, X) = [\tilde{f}_T(t, X), \hat{H}]. \quad (5.34)$$

The commutator in Eq. (5.34) may be expressed as

$$[\tilde{f}_T(t, X), \hat{H}] = \exp\left(\frac{i}{\hbar}\hat{H}t\right) [\hat{f}_T(X), \hat{H}] \exp\left(-\frac{i}{\hbar}\hat{H}t\right).$$

If the Hamiltonian has the form $\hat{H} = \frac{1}{2M}\hat{p}^2 + V(\hat{x})$, then the commutator takes the form

$$[\hat{f}_T(X), \hat{H}] = i\hbar \hat{J}(X) \quad (5.35)$$

where $\hat{J}(X)$ is the probability flux operator,

$$\hat{J}(x) = \frac{1}{2M} (|x\rangle\langle x|\hat{p} + \hat{p}|x\rangle\langle x|). \quad (5.36)$$

Therefore, we have an equation for the commutator

$$\left[\tilde{f}_T(t, X), \hat{H} \right] = i\hbar \tilde{J}(X, t). \quad (5.37)$$

The initial condition for the function $f(t, X)$ may be defined as

$$\tilde{f}_T(t = 0, X) = \hat{f}_T(X).$$

From Eqs. (5.34) and (5.37) we obtain the equation for the evolution of the tunneling flag operator

$$i\hbar \frac{\partial}{\partial t} \tilde{f}_T(t, X) = i\hbar \tilde{J}(X, t). \quad (5.38)$$

From Eq. (5.38) and the initial condition, an explicit expression for the tunneling flag operator follows

$$\tilde{f}_T(t, X) = \hat{f}_T(X) + \int_0^t \tilde{J}(X, t_1) dt_1. \quad (5.39)$$

In the question of *how much time does the tunneling particle spends under the barrier*, we ask about the particles, which we know with certainty have tunneled. In addition, we want to have some information about the location of the particle. However, does quantum mechanics allow us to have the information about the tunneling and location simultaneously? A projection operator

$$\hat{D}(\Gamma) = \int_{\Gamma} |x\rangle\langle x| dx \quad (5.40)$$

represents the probability for the particle to be in the region Γ . Here $|x\rangle$ is the eigenfunction of the coordinate operator. In Heisenberg representation this operator takes the form

$$\tilde{D}(\Gamma, t) = \exp\left(\frac{i}{\hbar} \hat{H} t\right) \hat{D}(\Gamma) \exp\left(-\frac{i}{\hbar} \hat{H} t\right). \quad (5.41)$$

From Eqs. (5.36), (5.39) and (5.41) we see that the operators $\tilde{D}(\Gamma, t)$ and $\tilde{f}_T(\infty, X)$ in general do not commute. This means that we cannot simultaneously have the information about the tunneling and location of the particle. If we know with certainty that the particle has tunneled then we can say nothing about its location in the past and if we know something about the location of the particle, we cannot determine definitely whether the particle will tunnel. Therefore, the question of *how much time does the tunneling particle spend under the barrier* cannot have definite answer, if the question is so posed that its precise definition requires the existence of the joint probability that the particle is found in Γ at time t and whether or not it is found on the right side of the barrier at a sufficiently later time. A similar analysis has been performed in Ref. [116]. It has been shown that due to noncommutability of operators there exist no unique decomposition of the dwell time.

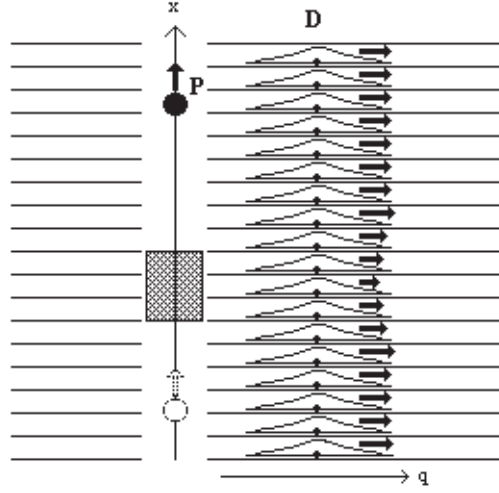


Figure 5.3: The configuration of the measurements of the tunneling time. The particle \mathbf{P} is tunneling along x coordinate and it is interacting with detectors \mathbf{D} . The barrier is represented by the rectangle. The interaction with the definite detector occurs only in the narrow region limited by the horizontal lines. The changes of the momenta of the detectors are represented by arrows.

This conclusion is, however, not only negative. We know that $\int_{-\infty}^{+\infty} |x\rangle\langle x| dx = 1$ and $[1, \tilde{f}_T(\infty, X)] = 0$. Therefore, if the region Γ is large enough, one has a possibility to answer the question about the tunneling time.

From the fact that the operators $\tilde{D}(\Gamma, t)$ and $\tilde{f}_T(\infty, X)$ do not commute we can predict that the measurement of the tunneling time will yield a value dependent on the particular detection scheme. The detector is made so that it yields some value. But if we try to measure noncommuting observables, the measured values depend on the interaction between the detector and the measured system. So, in the definition of the Larmor time there is a dependence on the type of boundary attributed to the magnetic-field region [82].

5.4.2 The model of the time measurement

We consider a model for the tunneling time measurement which is somewhat similar to the “gedanken” experiment used to obtain the Larmor time, but it is simpler and more transparent. This model had been proposed by Steinberg [104], however, it was treated in a nonstandard way, introducing complex probabilities. Here we use only the formalism of the standard quantum mechanics.

Our system consists of particle \mathbf{P} and several detectors \mathbf{D} [53]. Each detector interacts with the particle only in the narrow region of space. The configuration of the system is shown in Fig. 5.3. When the interaction of the particle with the detectors is weak, the detectors do not influence the state of the particle. Therefore, we can analyze the action of detectors separately. This model is a particular case of time measurement, presented in Sec. 5.3.1, with χ_D being the position of the detector x_D . Similar calculations were done

for detector's position rather than momentum by G. Iannaccone [117].

In the moment $t = 0$ the particle must be before the barrier, therefore, $\langle x | \rho_P(0) | x' \rangle \neq 0$ only when $x < 0$ and $x' < 0$, where $\hat{\rho}_P(0)$ is the density matrix of the particle.

5.4.3 Measurement of the dwell time

As in Sec. 5.3.2 we obtain the time the particle spends in the unit length region between time momentum $t = 0$ and t

$$\tau^{\text{Dw}}(x, t) = \langle \hat{F}(x, t) \rangle. \quad (5.42)$$

The time spent in the space region restricted by the coordinates x_1 and x_2 is

$$t^{\text{Dw}}(x_2, x_1) = \int_{x_1}^{x_2} \tau^{\text{Dw}}(x, t \rightarrow \infty) dx = \int_{x_1}^{x_2} dx \int_0^\infty \rho(x, t) dt. \quad (5.43)$$

This is a well-known expression for the dwell time [82]. The dwell time is the average over entire ensemble of particles regardless they are tunneled or not. The expression for the dwell time obtained in our model is the same as the well-known expression obtained by other authors. Therefore, we can expect that our model can yield a reasonable expression for the tunneling time as well.

5.4.4 Conditional probabilities and the tunneling time

Having seen that our model gives the time averaged over the entire ensemble of particles, let us now take the average only over the subensemble of the tunneled particles. This is done similarly to Sec. 5.3.3 with \hat{P}_t replaced by the tunneling flag operator $\hat{f}_T(X)$, defined by Eq. (5.32). From Eq. (5.15) we obtain the time the tunneled particle spends in the unit length region around x until time t [53]

$$\begin{aligned} \tau(x, t) &= \frac{1}{2\langle \tilde{f}_T(t, X) \rangle} \left\langle \tilde{f}_T(t, X) \hat{F}(x, t) + \hat{F}(x, t) \tilde{f}_T(t, X) \right\rangle \\ &+ \frac{1}{i\hbar\langle \tilde{f}_T(t, X) \rangle} (\langle q \rangle \langle p_q \rangle - \text{Re}\langle \hat{q} \hat{p}_q \rangle) \left\langle \left[\tilde{f}_T(t, X), \hat{F}(x, t) \right] \right\rangle. \end{aligned} \quad (5.44)$$

The obtained expression (5.44) for the tunneling time is real, contrary to the complex-time approach. It should be noted that this expression even in the limit of the very weak measurement depends on the particular detector. If the commutator $[\tilde{f}_T(t, X), \hat{F}(x, t)]$ is zero, the time has a precise value. If the commutator is not zero, only the integral of this expression over a large region has the meaning of an asymptotic time related to the large region as we will see in Sec. 5.4.7.

Equation (5.44) can be rewritten as a sum of two terms, the first term being independent of the detector and the second dependent, i.e.,

$$\tau(x, t) = \tau^{\text{Tun}}(x, t) + \frac{2}{\hbar} (\langle q \rangle \langle p_q \rangle - \text{Re}\langle \hat{q} \hat{p}_q \rangle) \tau_{\text{corr}}^{\text{Tun}}(x, t) \quad (5.45)$$

where

$$\tau^{\text{Tun}}(x, t) = \frac{1}{2\langle \tilde{f}_T(t, X) \rangle} \left\langle \tilde{f}_T(t, X) \hat{F}(x, t) + \hat{F}(x, t) \tilde{f}_T(t, X) \right\rangle, \quad (5.46)$$

$$\tau_{\text{corr}}^{\text{Tun}}(x, t) = \frac{1}{2i\langle \tilde{f}_T(t, X) \rangle} \left\langle \left[\tilde{f}_T(t, X), \hat{F}(x, t) \right] \right\rangle. \quad (5.47)$$

The quantities $\tau^{\text{Tun}}(x, t)$ and $\tau_{\text{corr}}^{\text{Tun}}(x, t)$ do not depend on the detector.

In order to separate the tunneled and reflected particles we have to take the limit $t \rightarrow \infty$. Otherwise, the particles that tunnel after the time t would not contribute to the calculation. So we introduce operators

$$\hat{F}(x) = \int_0^\infty \tilde{D}(x, t_1) dt_1, \quad (5.48)$$

$$\hat{N}(x) = \int_0^\infty \tilde{J}(x, t_1) dt_1. \quad (5.49)$$

From Eq. (5.39) it follows that the operator $\tilde{f}_T(\infty, X)$ is equal to $\hat{f}_T(X) + \hat{N}(X)$. As long as the particle is initially before the barrier

$$\hat{f}_T(X) \hat{\rho}_P(0) = \hat{\rho}_P(0) \hat{f}_T(X) = 0.$$

In the limit $t \rightarrow \infty$ we have

$$\tau^{\text{Tun}}(x) = \frac{1}{2\langle \hat{N}(X) \rangle} \left\langle \hat{N}(X) \hat{F}(x) + \hat{F}(x) \hat{N}(X) \right\rangle, \quad (5.50)$$

$$\tau_{\text{corr}}^{\text{Tun}}(x) = \frac{1}{2i\langle \hat{N}(X) \rangle} \left\langle \left[\hat{N}(X), \hat{F}(x) \right] \right\rangle. \quad (5.51)$$

Let us define an ‘‘asymptotic time’’ as the integral of $\tau(x, \infty)$ over a wide region containing the barrier. Since the integral of $\tau_{\text{corr}}^{\text{Tun}}(x)$ is very small compared to that of $\tau^{\text{Tun}}(x)$ as we will see later, the asymptotic time is effectively the integral of $\tau^{\text{Tun}}(x)$ only. This allows us to identify $\tau^{\text{Tun}}(x)$ as ‘‘the density of the tunneling time’’.

In many cases for the simplification of mathematics it is common to write the integrals over time as the integrals from $-\infty$ to $+\infty$. In our model we cannot without additional assumptions integrate in Eqs. (5.48), (5.49) from $-\infty$ because the negative time means the motion of the particle to the initial position. If some particles in the initial wave packet had negative momenta then in the limit $t \rightarrow -\infty$ it was behind the barrier and contributed to the tunneling time.

5.4.5 Properties of the tunneling time

As it has been mentioned above, the question of *how much time does a tunneling particle spends under the barrier* has no exact answer. We can determine only the time the tunneling particle spends in a large region containing the barrier. In our model this time is expressed as an integral of quantity (5.50) over the region. In order to determine the properties of this integral it is useful to determine properties of the integrand.

To be able to expand the range of integration over time to $-\infty$, it is necessary to have the initial wave packet far to the left from the points under the investigation and this wave packet must consist only of the waves moving in the positive direction.

It is convenient to make calculations in the energy representation. Eigenfunctions of the Hamiltonian \hat{H}_P are $|E, \alpha\rangle$, where $\alpha = \pm 1$. The sign '+' or '-' corresponds to the positive or negative initial direction of the wave, respectively. Outside the barrier these eigenfunctions are

$$\langle x|E, +\rangle = \begin{cases} \sqrt{\frac{M}{2\pi\hbar p_E}} (\exp(\frac{i}{\hbar}p_E x) + r(E) \exp(-\frac{i}{\hbar}p_E x)), & x < 0, \\ \sqrt{\frac{M}{2\pi\hbar p_E}} t(E) \exp(\frac{i}{\hbar}p_E x), & x > L, \end{cases} \quad (5.52)$$

$$\langle x|E, -\rangle = \begin{cases} \sqrt{\frac{M}{2\pi\hbar p_E}} t(E) \exp(-\frac{i}{\hbar}p_E x), & x < 0, \\ \sqrt{\frac{M}{2\pi\hbar p_E}} \left(\exp(-\frac{i}{\hbar}p_E x) - \frac{t(E)}{t^*(E)} r^*(E) \exp(\frac{i}{\hbar}p_E x) \right), & x > L \end{cases} \quad (5.53)$$

where $t(E)$ and $r(E)$ are transmission and reflection amplitudes respectively,

$$p_E = \sqrt{2ME}, \quad (5.54)$$

the barrier is in the region between $x = 0$ and $x = L$ and M is the mass of the particle. These eigenfunctions are orthonormal, i.e.,

$$\langle E, \alpha|E', \alpha'\rangle = \delta_{\alpha, \alpha'} \delta(E - E'). \quad (5.55)$$

The evolution operator is

$$\hat{U}_P(t) = \sum_{\alpha} \int_0^{\infty} |E, \alpha\rangle \langle E, \alpha| \exp\left(-\frac{i}{\hbar}Et\right) dE.$$

The operator $\hat{F}(x)$ is given by the equation

$$\hat{F}(x) = \int_{-\infty}^{\infty} dt_1 \sum_{\alpha, \alpha'} \iint dE dE' |E, \alpha\rangle \langle E, \alpha|x\rangle \langle x|E', \alpha'\rangle \langle E', \alpha'| \exp\left(\frac{i}{\hbar}(E - E')t_1\right)$$

where the integral over the time is $2\pi\hbar\delta(E - E')$ and, therefore,

$$\hat{F}(x) = 2\pi\hbar \sum_{\alpha, \alpha'} \int dE |E, \alpha\rangle \langle E, \alpha|x\rangle \langle x|E, \alpha'\rangle \langle E, \alpha'|.$$

In an analogous way

$$\hat{N}(x) = 2\pi\hbar \sum_{\alpha, \alpha'} \int dE |E, \alpha\rangle \langle E, \alpha|\hat{J}(x)|E, \alpha'\rangle \langle E, \alpha'|.$$

We consider the initial wave packet consisting only of the waves moving in the positive direction. Then we have

$$\begin{aligned} \langle \hat{N}(x) \rangle &= 2\pi\hbar \int dE \langle |E, +\rangle \langle E, +|\hat{J}(x)|E, +\rangle \langle E, +| \rangle, \\ \langle \hat{F}(x)\hat{N}(X) \rangle &= 4\pi^2\hbar^2 \sum_{\alpha} \int dE \langle |E, +\rangle \langle E, +|x\rangle \langle x|E, \alpha\rangle \langle E, \alpha|\hat{J}(X)|E, +\rangle \langle E, +| \rangle. \end{aligned}$$

From the condition $X > L$ it follows

$$\langle \hat{N}(X) \rangle = \int dE \langle |E, + \rangle |t(E)|^2 \langle E, + | \rangle. \quad (5.56)$$

For $x < 0$ we obtain the following expressions for the quantities $\tau^{\text{Tun}}(x, t)$ and $\tau_{\text{corr}}^{\text{Tun}}(x, t)$

$$\begin{aligned} \tau^{\text{Tun}}(x, t) &= \frac{M}{\langle \hat{N}(X) \rangle} \int dE \left\langle |E, + \rangle \frac{1}{2p_E} |t(E)|^2 \left(2 + r(E) \exp \left(-2 \frac{i}{\hbar} p_E x \right) \right. \right. \\ &\quad \left. \left. + r^*(E) \exp \left(2 \frac{i}{\hbar} p_E x \right) \right) \langle E, + | \right\rangle, \end{aligned} \quad (5.57)$$

$$\begin{aligned} \tau_{\text{corr}}^{\text{Tun}}(x, t) &= \frac{M}{2\langle \hat{N}(X) \rangle} \int dE \left\langle |E, + \rangle \frac{1}{ip_E} |t(E)|^2 \left(r(E) \exp \left(-2 \frac{i}{\hbar} p_E x \right) \right. \right. \\ &\quad \left. \left. - r^*(E) \exp \left(2 \frac{i}{\hbar} p_E x \right) \right) \langle E, + | \right\rangle. \end{aligned} \quad (5.58)$$

For $x > L$ these expressions take the form

$$\begin{aligned} \tau^{\text{Tun}}(x, t) &= \frac{M}{\langle \hat{N}(X) \rangle} \int dE \left\langle |E, + \rangle \frac{1}{2p_E} |t(E)|^2 \left(2 - \frac{t(E)}{t^*(E)} r^*(E) \exp \left(2 \frac{i}{\hbar} p_E x \right) \right. \right. \\ &\quad \left. \left. - \frac{t^*(E)}{t(E)} r(E) \exp \left(-2 \frac{i}{\hbar} p_E x \right) \right) \langle E, + | \right\rangle, \end{aligned} \quad (5.59)$$

$$\begin{aligned} \tau_{\text{corr}}^{\text{Tun}}(x, t) &= \frac{M}{2\langle \hat{N}(X) \rangle} \int dE \left\langle |E, + \rangle \frac{i}{p_E} |t(E)|^2 \left(\frac{t(E)}{t^*(E)} r^*(E) \exp \left(2 \frac{i}{\hbar} p_E x \right) \right. \right. \\ &\quad \left. \left. - \frac{t^*(E)}{t(E)} r(E) \exp \left(-2 \frac{i}{\hbar} p_E x \right) \right) \langle E, + | \right\rangle. \end{aligned} \quad (5.60)$$

We illustrate the obtained formulae for the δ -function barrier

$$V(x) = \Omega \delta(x)$$

and for the rectangular barrier. The incident wave packet is Gaussian and it is localized far to the left from the barrier.

In Fig. 5.4 and 5.5, we see interferencelike oscillations near the barrier. Oscillations are not only in front of the barrier but also behind the barrier. When x is far from the barrier the “time density” tends to a value close to 1. This is in agreement with classical mechanics because in the chosen units the mean velocity of the particle is 1. In Fig. 5.5, another property of “tunneling time density” is seen: it is almost zero in the barrier region. This explains the Hartmann and Fletcher effect [118, 119]: for opaque barriers the effective tunneling velocity is very large.

5.4.6 The reflection time

We can easily adapt our model for the reflection too. For doing this, we should replace the tunneling-flag operator \hat{f}_T by the reflection flag operator

$$\hat{f}_R = 1 - \hat{f}_T. \quad (5.61)$$

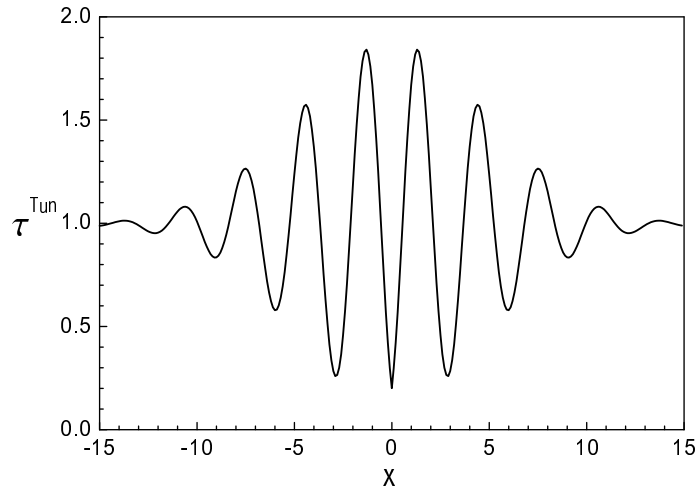


Figure 5.4: The asymptotic time density for δ -function barrier with the parameter $\Omega = 2$. The barrier is located at the point $x = 0$. The units are such that $\hbar = 1$ and $M = 1$ and the average momentum of the Gaussian wave packet $\langle p \rangle = 1$. In these units length and time are dimensionless. The width of the wave packet in the momentum space $\sigma = 0.001$.

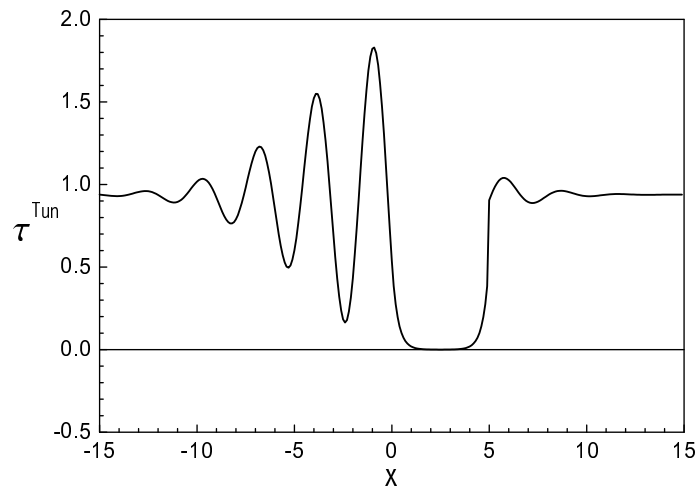


Figure 5.5: The asymptotic time density for rectangular barrier. The barrier is localized between the points $x = 0$ and $x = 5$ and the height of the barrier is $V_0 = 2$. The used units and parameters of the initial wave packet are the same as in Fig. 5.4.

Replacing \hat{f}_T by \hat{f}_R in Eqs. (5.50) and (5.51) we obtain the equality

$$\langle \tilde{f}_R(t = \infty, X) \rangle \tau^{\text{Refl}}(x) = \tau^{\text{Dw}}(x) - \langle \tilde{f}_T(t = \infty, X) \rangle \tau^{\text{Tun}}(x). \quad (5.62)$$

We see that in our model the important condition

$$\tau^{\text{Dw}} = T\tau^{\text{Tun}} + R\tau^{\text{Refl}} \quad (5.63)$$

where T and R are transmission and reflection probabilities is satisfied automatically.

If the wave packet consists of only the waves moving in the positive direction, the density of dwell time is

$$\tau^{\text{Dw}}(x, t) = 2\pi\hbar \int dE \langle |E, + \rangle \langle E, + | x \rangle \langle x | E, + \rangle \langle E, + | \rangle. \quad (5.64)$$

For $x < 0$ we have

$$\begin{aligned} \tau^{\text{Dw}}(x, t) = M \int dE \left\langle |E, + \rangle \frac{1}{p_E} \left(1 + |r(E)|^2 + r(E) \exp\left(-2\frac{i}{\hbar}p_E x\right) \right. \right. \\ \left. \left. + r^*(E) \exp\left(2\frac{i}{\hbar}p_E x\right) \right) \langle E, + | \right\rangle \end{aligned} \quad (5.65)$$

and for the reflection time we obtain the “time density”

$$\begin{aligned} \tau^{\text{Refl}}(x) = \frac{M}{1 - \langle \hat{N}(X) \rangle} \int dE \left\langle |E, + \rangle \frac{1}{p_E} (2|r(E)|^2 \right. \\ \left. + \frac{1}{2} (1 + |r(E)|^2) r(E) \exp\left(-2\frac{i}{\hbar}p_E x\right) + r^*(E) \exp\left(2\frac{i}{\hbar}p_E x\right) \right) \langle E, + | \right\rangle. \end{aligned} \quad (5.66)$$

For $x > L$ the density of the dwell time is

$$\tau^{\text{Dw}}(x, t) = M \int dE \left\langle |E, + \rangle \frac{1}{p_E} |t(E)|^2 \langle E, + | \right\rangle \quad (5.67)$$

and the “density of the reflection time” may be expressed as

$$\begin{aligned} \tau^{\text{Refl}}(x) = \frac{M}{2} \int dE \left\langle |E, + \rangle \frac{1}{p_E} |t(E)|^2 \left(\frac{t(E)}{t^*(E)} r^*(E) \exp\left(2\frac{i}{\hbar}p_E x\right) \right. \right. \\ \left. \left. + \frac{t^*(E)}{t(E)} r(E) \exp\left(-2\frac{i}{\hbar}p_E x\right) \right) \langle E, + | \right\rangle. \end{aligned} \quad (5.68)$$

We illustrate the properties of the reflection time for the same barriers. The incident wave packet is Gaussian and it is localized far to the left from the barrier. In Figs. 5.6 and 5.7, we also see the interferencelike oscillations at both sides of the barrier. As far as for the rectangular barrier the “time density” is very small, the part behind the barrier is presented in Fig. 5.8. Behind the barrier, the “time density” in certain places becomes negative. This is because the quantity $\tau^{\text{Refl}}(x)$ is not positive definite. Nonpositivity is the direct consequence of noncommutativity of operators in Eqs. (5.50) and (5.51). There is nothing strange in the negativity of $\tau^{\text{Refl}}(x)$ because this quantity itself has no physical meaning. Only the integral over the large region has the meaning of time. When x is far to the left from the barrier the “time density” tends to a value close to 2 and when x is far to the right from the barrier the “time density” tends to 0. This is in agreement with classical mechanics because in the chosen units, the velocity of the particle is 1 and the reflected particle crosses the area before the barrier two times.

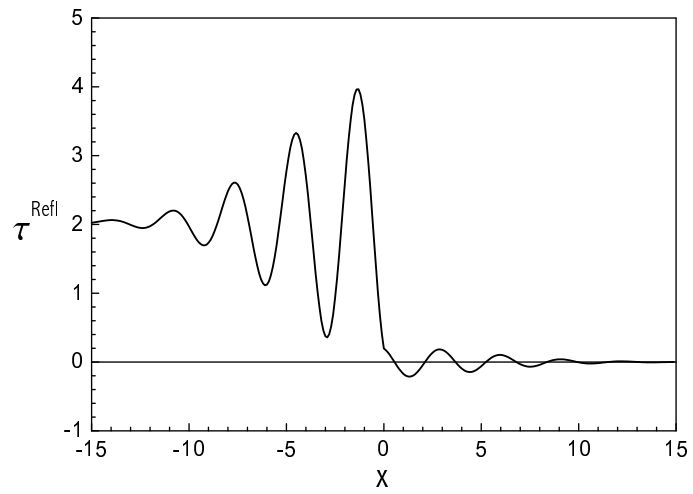


Figure 5.6: Reflection time density for the same conditions as in Fig. 5.4.

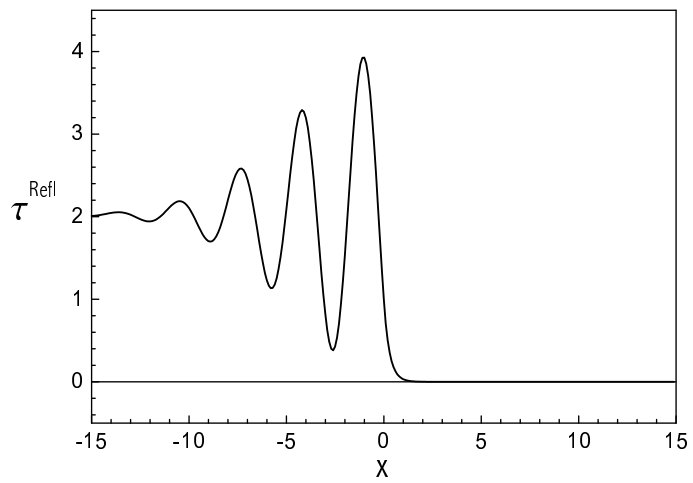


Figure 5.7: Reflection time density for the same conditions as in Fig. 5.5.

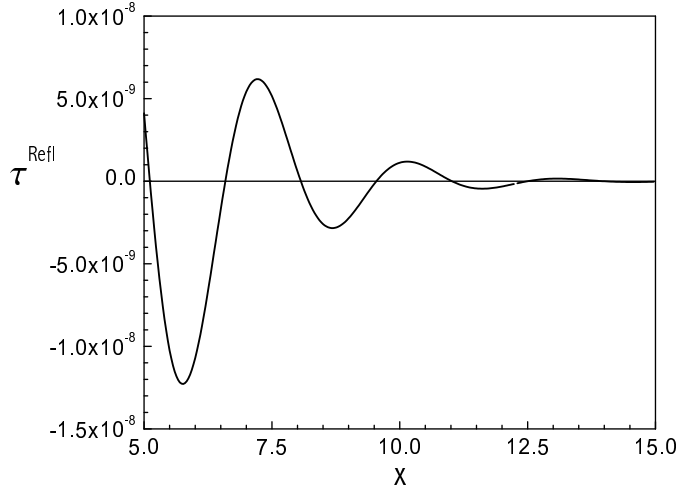


Figure 5.8: Reflection time density for rectangular barrier in the area behind the barrier. The parameters and the initial conditions are the same as in Fig. 5.5

5.4.7 The asymptotic time

As mentioned above, we can determine only the time that the tunneling particle spends in a large region containing the barrier, i.e., the asymptotic time. In our model this time is expressed as an integral of quantity (5.50) over this region. We can do the integration explicitly.

The continuity equation yields

$$\frac{\partial}{\partial t} \tilde{D}(x_D, t) + \frac{\partial}{\partial x} \tilde{J}(x_D, t) = 0. \quad (5.69)$$

The integration can be performed by parts

$$\int_0^t \tilde{D}(x_D, t_1) dt_1 = t \tilde{D}(x_D, t) + \frac{\partial}{\partial x} \int_0^t t_1 \tilde{J}(x_D, t_1) dt_1.$$

If the density matrix $\hat{\rho}_P(0)$ represents localized particle then $\lim_{t \rightarrow \infty} \left(\tilde{D}(x, t) \hat{\rho}_P(0) \right) = 0$. Therefore we can write an effective equality

$$\int_0^\infty \tilde{D}(x_D, t_1) dt_1 = \frac{\partial}{\partial x} \int_0^\infty t_1 \tilde{J}(x_D, t_1) dt_1. \quad (5.70)$$

We introduce the operator

$$\hat{T}(x) = \int_0^\infty t_1 \tilde{J}(x, t_1) dt_1. \quad (5.71)$$

We consider the asymptotic time, i.e., the time the particle spends between points x_1 and x_2 when $x_1 \rightarrow -\infty$, $x_2 \rightarrow +\infty$,

$$t^{\text{Tun}}(x_2, x_1) = \int_{x_1}^{x_2} \tau^{\text{Tun}}(x) dx.$$

After the integration we have

$$t^{\text{Tun}}(x_2, x_1) = t^{\text{Tun}}(x_2) - t^{\text{Tun}}(x_1) \quad (5.72)$$

where

$$t^{\text{Tun}}(x) = \frac{1}{2\langle \hat{N}(x) \rangle} \left\langle \hat{N}(x) \hat{T}(x) + \hat{T}(x) \hat{N}(x) \right\rangle. \quad (5.73)$$

If we assume that the initial wave packet is far to the left from the points under the investigation and consists only of the waves moving in the positive direction, then Eq. (5.72) may be simplified.

In the energy representation

$$\hat{T}(x) = \int_{-\infty}^{\infty} t_1 dt_1 \sum_{\alpha, \alpha'} \iint dE dE' |E, \alpha\rangle \langle E, \alpha | \hat{J}(x) |E', \alpha'\rangle \langle E', \alpha' | \exp\left(\frac{i}{\hbar}(E - E')t_1\right).$$

The integral over time is equal to $2i\pi\hbar^2 \frac{\partial}{\partial E'} \delta(E - E')$ and we obtain

$$\begin{aligned} \hat{T}(x) &= -i\hbar 2\pi\hbar \sum_{\alpha, \alpha'} \int dE |E, \alpha\rangle \left(\frac{\partial}{\partial E'} \langle E, \alpha | \hat{J}(x) |E', \alpha'\rangle \Big|_{E'=E} \langle E, \alpha' | \right. \\ &\quad \left. + \langle E, \alpha | \hat{J}(x) |E, \alpha'\rangle \frac{\partial}{\partial E} \langle E, \alpha' | \right), \\ \langle \hat{N}(X) \hat{T}(x) \rangle &= -i\hbar 4\pi^2 \hbar^2 \sum_{\alpha} \int dE \langle \Psi | E, + \rangle \langle E, + | \hat{J}(X) | E, \alpha \rangle \\ &\quad \times \left(\frac{\partial}{\partial E'} \langle E, \alpha | \hat{J}(x) | E', + \rangle \Big|_{E'=E} + \langle E, \alpha | \hat{J}(x) | E, + \rangle \frac{\partial}{\partial E} \right) \langle E, + | \Psi \rangle. \end{aligned}$$

Substituting expressions for the matrix elements of the probability flux operator we obtain equation

$$\begin{aligned} \langle \hat{N}(X) \hat{T}(x) \rangle &= \int dE \langle \Psi | E, + \rangle t^*(E) \frac{\hbar}{i} \frac{\partial}{\partial E} t(E) \langle E, + | \Psi \rangle \\ &\quad + Mx \int dE \langle \Psi | E, + \rangle \frac{1}{p_E} |t(E)|^2 \langle E, + | \Psi \rangle \\ &\quad + i\hbar \frac{M}{2} \int dE \langle \Psi | E, + \rangle \frac{1}{p_E^2} r^*(E) t^2(E) \exp\left(2\frac{i}{\hbar} p_E x\right) \langle E, + | \Psi \rangle. \end{aligned}$$

When $x \rightarrow +\infty$, the last term vanishes and we have

$$\begin{aligned} \langle \hat{N}(X) \hat{T}(x) \rangle &= \int dE \langle \Psi | E, + \rangle t^*(E) \frac{\hbar}{i} \frac{\partial}{\partial E} t(E) \langle E, + | \Psi \rangle \\ &\quad + Mx \int dE \langle \Psi | E, + \rangle \frac{1}{p_E} |t(E)|^2 \langle E, + | \Psi \rangle, \quad x \rightarrow +\infty. \end{aligned} \quad (5.74)$$

This expression is equal to $\langle \hat{T}(x) \rangle$,

$$\langle \hat{N}(X) \hat{T}(x) \rangle \rightarrow \langle \hat{T}(x) \rangle, \quad x \rightarrow +\infty. \quad (5.75)$$

When the point with coordinate x is in front of the barrier, we obtain an equality

$$\begin{aligned} \langle \hat{N}(X)\hat{T}(x) \rangle &= -i\hbar \int dE \langle \Psi|E, + \rangle |t(E)|^2 \left(\frac{i}{\hbar} \frac{M}{p_E} x \right. \\ &\quad \left. - \frac{M}{2p_E^2} r(E) \exp\left(-\frac{i}{\hbar} 2p_E x\right) + \frac{\partial}{\partial E} \right) \langle E, + | \Psi \rangle \end{aligned}$$

When $|x|$ is large the second term vanishes and we have

$$\begin{aligned} \langle \hat{N}(X)\hat{T}(x) \rangle &\rightarrow Mx \int dE \langle \Psi|E, + \rangle \frac{1}{p_E} |t(E)|^2 \langle E, + | \Psi \rangle \\ &\quad + \int dE \langle \Psi|E, + \rangle |t(E)|^2 \frac{\hbar}{i} \frac{\partial}{\partial E} \langle E, + | \Psi \rangle. \end{aligned} \quad (5.76)$$

The imaginary part of expression (5.76) is not zero. This means that for determination of the asymptotic time it is insufficient to integrate only in the region containing the barrier. For quasimonochromatic wave packets from Eqs. (5.71), (5.72), (5.73), (5.74) and (5.76) we obtain limits

$$t^{\text{Tun}}(x_2, x_1) \rightarrow t_T^{\text{Ph}} + \frac{1}{p_E} M(x_2 - x_1), \quad (5.77)$$

$$t_{\text{corr}}^{\text{Tun}}(x_2, x_1) \rightarrow -t_T^{\text{Im}} \quad (5.78)$$

where

$$t_T^{\text{Ph}} = \hbar \frac{d}{dE} (\arg t(E)) \quad (5.79)$$

is the phase time and

$$t_T^{\text{Im}} = \hbar \frac{d}{dE} (\ln |t(E)|) \quad (5.80)$$

is the imaginary part of the complex time.

In order to take the limit $x \rightarrow -\infty$ we have to perform more exact calculations. We cannot extend the range of the integration over the time to $-\infty$ because this extension corresponds to the initial wave packet being infinitely far from the barrier. We can extend the range of the integration over the time to $-\infty$ only for calculation of $\hat{N}(X)$. For $x < 0$ we obtain the following equality

$$\langle \hat{N}(X)\hat{T}(x) \rangle = \frac{1}{4\pi M i} \int_0^\infty t dt \left(I_1^*(x, t) \frac{\partial}{\partial x} I_2(x, t) - I_2(x, t) \frac{\partial}{\partial x} I_1^*(x, t) \right) \quad (5.81)$$

where

$$I_1(x, t) = \int dE \frac{1}{\sqrt{p_E}} |t(E)|^2 \exp\left(\frac{i}{\hbar} (p_E x - Et)\right) \langle E, + | \Psi \rangle, \quad (5.82)$$

$$I_2(x, t) = \int dE \frac{1}{\sqrt{p_E}} \left(\exp\left(\frac{i}{\hbar} p_E x\right) + r(E) \exp\left(-\frac{i}{\hbar} p_E x\right) \right) \exp\left(-\frac{i}{\hbar} Et\right) \langle E, + | \Psi \rangle. \quad (5.83)$$

$I_1(x, t)$ is equal to the wave function in the point x at the time moment t when the propagation is in the free space and the initial wave function in the energy representation

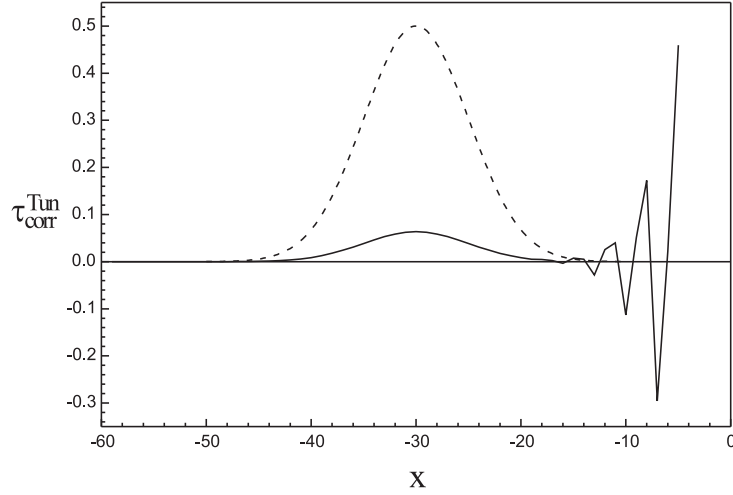


Figure 5.9: The quantity $\tau_{\text{corr}}^{\text{Tun}}(x)$ for δ function barrier with the parameters and initial conditions as in Fig. 5.4. The initial packet is shown with dashed line.

is $|t(E)|^2 \langle E, + | \Psi \rangle$. When $t \geq 0$ and $x \rightarrow -\infty$, then $I_1(x, t) \rightarrow 0$. That is why the initial wave packet contains only the waves moving in the positive direction. Therefore $\langle \hat{N}(X) \hat{T}(x) \rangle \rightarrow 0$ when $x \rightarrow -\infty$. From this analysis it follows that the region in which the asymptotic time is determined has to contain not only the barrier but also the initial wave packet.

In such a case from Eqs. (5.72) and (5.73) we obtain expression for the asymptotic time

$$t^{\text{Tun}}(x_2, x_1 \rightarrow -\infty) = \frac{1}{\langle \hat{N}(X) \rangle} \int dE \langle \Psi | E, + \rangle t^*(E) \left(\frac{M}{pE} x_2 - i\hbar \frac{\partial}{\partial E} \right) t(E) \langle E, + | \Psi \rangle. \quad (5.84)$$

From Eq. (5.75) it follows that

$$t^{\text{Tun}}(x_2, x_1 \rightarrow -\infty) = \frac{1}{\langle \hat{N}(X) \rangle} \langle \hat{T}(x_2) \rangle \quad (5.85)$$

where $\hat{T}(x_2)$ is defined as the probability flux integral (5.71). Equations (5.84) and (5.85) give the same value for tunneling time as an approach in Refs. [120, 121]

The integral of quantity $\tau_{\text{corr}}^{\text{Tun}}(x)$ over a large region is zero. We have seen that it is not enough to choose the region around the barrier—this region has to include also the initial wave packet location. We illustrate this fact by numerical calculations.

The quantity $\tau_{\text{corr}}^{\text{Tun}}(x)$ for δ -function barrier is represented in Fig. 5.9. We see that $\tau_{\text{corr}}^{\text{Tun}}(x)$ is not equal to zero not only in the region around the barrier but also it is not zero in the location of the initial wave packet. For comparison, the quantity $\tau^{\text{Tun}}(x)$ for the same conditions is represented in Fig. 5.10.

5.5 Weak measurement of arrival time

The detection of the particles in time-of-flight and coincidence experiments are common, and quantum mechanics should give a method for the calculation of the arrival time. The

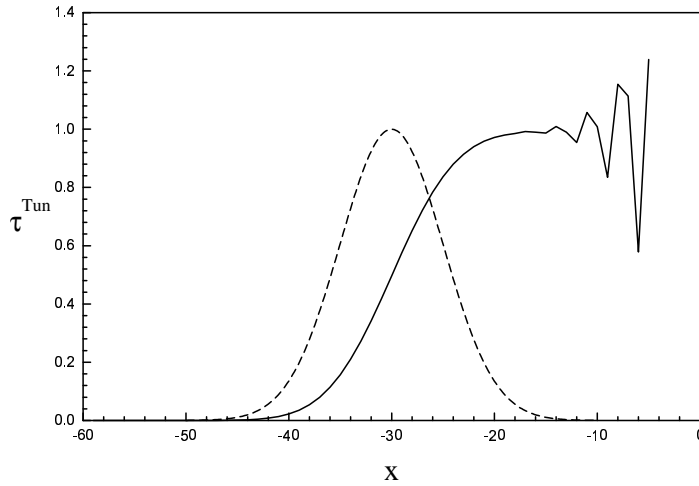


Figure 5.10: Tunneling time density for the same conditions and parameters as in Fig. 5.9.

arrival time distribution may be useful in solving the tunneling time problem, as well. Therefore, the quantum description of arrival time has attracted much attention [120–130].

Aharonov and Bohm introduced the arrival time operator [122]

$$\hat{T}_{AB} = \frac{m}{2} \left((X - \hat{x}) \frac{1}{\hat{p}} + \frac{1}{\hat{p}} (X - \hat{x}) \right). \quad (5.86)$$

By imposing several conditions (normalization, positivity, minimum variance, and symmetry with respect to the arrival point X) a quantum arrival time distribution for a free particle was obtained by Kijowski [123]. Kijowski's distribution may be associated with the positive operator valued measure generated by the eigenstates of \hat{T}_{AB} . However, Kijowski's set of conditions cannot be applied in a general case [123]. Nevertheless, arrival time operators can be constructed even if the particle is not free [131, 132].

Since the mean arrival time even in classical mechanics can be infinite or the particle may not arrive at all, it is convenient to deal not with the mean arrival time and corresponding operator \hat{T} , but with the probability distribution of the arrival time [101]. The probability distribution of the arrival time can be obtained from a suitable classical definition. The noncommutativity of the operators in quantum mechanics is circumvented by using the concept of weak measurements.

5.5.1 Arrival time in classical mechanics

In classical mechanics the particle moves along the trajectory $H(x, p) = \text{const}$ as t increases. This allows us to work out the time of arrival at the point $x(t) = X$, by identifying the point (x_0, p_0) of the phase space where the particle is at $t = 0$, and then following the trajectory that passes by this point, up to arrival at the point X . If multiple crossings are possible, one may define a distribution of arrival times with contributions from all crossings, when no distinction is made between first, second and n th arrivals. In this article we will consider such a distribution.

We can ask whether there is a definition of the arrival time that is valid in both classical and quantum mechanics. In our opinion, the words “the particle arrives from the left at the point X at the time t ” mean that: (i) at time t the particle was in the region $x < X$ and (ii) at time $t + \Delta t$ ($\Delta t \rightarrow 0$) the particle is found in the region $x > X$. Now we apply this definition, given by (i) and (ii), to the time of arrival in the classical case.

Since quantum mechanics deals with probabilities, it is convenient to use probabilistic description of the classical mechanics, as well. Therefore, we will consider an ensemble of noninteracting classical particles. The probability density in the phase space is $\rho(x, p; t)$.

Let us denote the region $x < X$ as Γ_1 and the region $x > X$ as Γ_2 . The probability that the particle arrives from region Γ_1 to region Γ_2 at a time between t and $t + \Delta t$ is proportional to the probability that the particle is in region Γ_1 at time t and in region Γ_2 at time $t + \Delta t$. This probability is

$$\Pi_+(t)\Delta t = \frac{1}{N_+} \int_{\Omega} dp dx \rho(x, p; t), \quad (5.87)$$

where N_+ is the constant of normalization and the region of phase space Ω has the following properties: (i) the coordinates of the points in Ω are in the space region Γ_1 and (ii) if the phase trajectory goes through a point of the region Ω at time t then the particle at time $t + \Delta t$ is in the space region Γ_2 . Since Δt is infinitesimal, the change of coordinate during the time interval Δt is equal to $\frac{p}{m}\Delta t$. Therefore, the particle arrives from region Γ_1 to region Γ_2 only if the momentum of the particle at the point X is positive. The phase space region Ω consists of the points with positive momentum p and with coordinates between $X - p/m\Delta t$ and X . Then from Eq. (5.87) we have the probability of arrival time

$$\Pi_+(t)\Delta t = \frac{1}{N_+} \int_0^{\infty} dp \int_{X - \frac{p}{m}\Delta t}^X dx \rho(x, p; t). \quad (5.88)$$

Since Δt is infinitesimal and the momentum of every particle is finite, we can replace x in Eq. (5.88) by X and obtain the equality

$$\Pi_+(t, X) = \frac{1}{N_+} \int_0^{\infty} \frac{p}{m} \rho(X, p; t) dp. \quad (5.89)$$

The obtained arrival time distribution $\Pi_+(t, X)$ is well known and has appeared quite often in the literature (see, e.g., the review [131] and references therein).

The probability current in classical mechanics is

$$J(x; t) = \int_{-\infty}^{+\infty} \frac{p}{m} \rho(x, p; t) dp. \quad (5.90)$$

From Eqs. (5.89) and (5.90) it is clear that the time of arrival is related to the probability current. This relation, however, is not straightforward. We can introduce the “positive probability current”

$$J_+(x; t) = \int_0^{\infty} \frac{p}{m} \rho(x, p; t) dp \quad (5.91)$$

and rewrite Eq. (5.89) as

$$\Pi_+(t, X) = \frac{1}{N_+} J_+(X; t). \quad (5.92)$$

The proposed [133–135] various quantum versions of J_+ even in the case of the free particle can be negative (the so-called backflow effect). Therefore, the classical expression (5.92) for the time of arrival becomes problematic in quantum mechanics.

Similarly, for arrival from the right we obtain the probability density

$$\Pi_-(t, X) = \frac{1}{N_-} J_-(X; t), \quad (5.93)$$

where the negative probability current is

$$J_-(x; t) = \int_{-\infty}^0 \frac{|p|}{m} \rho(x, p; t) dp. \quad (5.94)$$

We see that our definition, given at the beginning of this section, leads to the proper result in classical mechanics. The conditions (i) and (ii) does not involve the concept of the trajectories. We can try to use this definition also in quantum mechanics.

5.5.2 Weak measurement of arrival time

The proposed definition of the arrival time probability distribution can be used in quantum mechanics only if the determination of the region in which the particle is does not disturb the motion of the particle. This can be achieved using the weak measurements of Aharonov, Albert and Vaidman [94–99].

We use the weak measurement, described in Sec. 5.2. The detector interacts with the particle only in region Γ_1 . As the operator \hat{A} we take the the projection operator \hat{P}_1 , projecting into region Γ_1 . In analogy to Ref. [95], we define the “weak value” of the probability of finding the particle in the region Γ_1 ,

$$W(1) \equiv \langle \hat{P}_1 \rangle = \frac{\langle \hat{p}_q \rangle_0 - \langle \hat{p}_q \rangle}{\lambda \tau}. \quad (5.95)$$

In order to obtain the arrival time probability using the definition from Sec. 5.5.1, we measure the momenta p_q of each detector after the interaction with the particle. After time Δt we perform the final, postselection measurement on the particles of our ensemble and measure if the particle is found in region Γ_2 . Then we collect the outcomes p_q only for the particles found in region Γ_2 .

The projection operator projecting into the region Γ_2 is \hat{P}_2 . In the Heisenberg representation this operator is

$$\tilde{P}_2(t) = \hat{U}(t)^\dagger \hat{P}_2 \hat{U}(t), \quad (5.96)$$

where \hat{U} is the evolution operator of the free particle. Taking the operator B from Sec. 5.2 as $\tilde{P}_2(\Delta t)$ and using Eq. (5.95) we can introduce the weak value $W(1|2)$ of probability to find the particle in region Γ_1 on condition that the particle after time Δt is in region Γ_2 . The probability that the particle is in region Γ_1 and after time Δt is in region Γ_2 equals to

$$W(1, 2) = W(2)W(1|2). \quad (5.97)$$

When the measurement time τ is sufficiently small, the influence of the Hamiltonian of the particle can be neglected. Using Eq. (5.7) from Sec. 5.2 we obtain

$$W(1, 2) \approx \frac{1}{2} \langle \tilde{P}_2(\Delta t) \hat{P}_1 + \hat{P}_1 \tilde{P}_2(\Delta t) \rangle + \frac{i}{\hbar} (\langle \hat{p}_q \rangle \langle \hat{q} \rangle - \text{Re} \langle \hat{q} \hat{p}_q \rangle) \langle [\hat{P}_1, \tilde{P}_2(\Delta t)] \rangle. \quad (5.98)$$

The probability $W(1, 2)$ is constructed using conditions (i) and (ii) from Sec. 5.5.1: the weak measurement is performed to determine if the particle is in region Γ_1 and after time Δt the strong measurement determines if the particle is in region Γ_2 . Therefore, according to Sec. 5.5.1, the quantity $W(1, 2)$ after normalization can be considered as the weak value of the arrival time probability distribution.

Equation (5.98) consists of two terms and we can introduce two quantities

$$\Pi^{(1)} = \frac{1}{2\Delta t} \langle \hat{P}_1 \tilde{P}_2(\Delta t) + \tilde{P}_2(\Delta t) \hat{P}_1 \rangle \quad (5.99)$$

and

$$\Pi^{(2)} = \frac{1}{2i\Delta t} \langle [\hat{P}_1, \tilde{P}_2(\Delta t)] \rangle. \quad (5.100)$$

Then

$$W(1, 2) = \Pi^{(1)} \Delta t - \frac{2\Delta t}{\hbar} (\langle \hat{p}_q \rangle \langle \hat{q} \rangle - \text{Re} \langle \hat{q} \hat{p}_q \rangle) \Pi^{(2)}. \quad (5.101)$$

If the commutator $[\hat{P}_1, \tilde{P}_2(\Delta t)]$ in Eqs. (5.99)–(5.101) is not zero, then, even in the limit of the very weak measurement, the measured value depends on the particular detector. This fact means that in such a case we cannot obtain a definite value for the arrival time probability. Moreover, the coefficient $(\langle \hat{p}_q \rangle \langle \hat{q} \rangle - \text{Re} \langle \hat{q} \hat{p}_q \rangle)$ may be zero for a specific initial state of the detector, e.g., for a Gaussian distribution of the coordinate q and momentum p_q .

The quantities $W(1, 2)$, $\Pi^{(1)}$ and $\Pi^{(2)}$ are real. However, it is convenient to consider the complex quantity

$$\Pi_C = \Pi^{(1)} + i\Pi^{(2)} = \frac{1}{\Delta t} \langle \hat{P}_1 \tilde{P}_2(\Delta t) \rangle. \quad (5.102)$$

We call it the “complex arrival probability”. We can introduce the corresponding operator

$$\hat{\Pi}_+ = \frac{1}{\Delta t} \hat{P}_1 \tilde{P}_2(\Delta t). \quad (5.103)$$

By analogy, the operator

$$\hat{\Pi}_- = \frac{1}{\Delta t} \tilde{P}_2 \hat{P}_1(\Delta t) \quad (5.104)$$

corresponds to arrival from the right.

The introduced operator $\hat{\Pi}_+$ has some properties of the classical positive probability current. From the conditions $\hat{P}_1 + \hat{P}_2 = 1$ and $\tilde{P}_1(\Delta t) + \tilde{P}_2(\Delta t) = 1$ we have

$$\hat{\Pi}_+ - \hat{\Pi}_- = \frac{1}{\Delta t} (\tilde{P}_2(\Delta t) - \hat{P}_2).$$

In the limit $\Delta t \rightarrow 0$ we obtain the probability current $\hat{J} = \lim_{\Delta t \rightarrow 0} (\hat{\Pi}_+ - \hat{\Pi}_-)$, as in classical mechanics. However, the quantity $\langle \hat{\Pi}_+ \rangle$ is complex and the real part can be negative, in contrast to the classical quantity J_+ . The reason for this is the noncommutativity of the operators \hat{P}_1 and $\tilde{P}_2(\Delta t)$. When the imaginary part is small, the quantity $\langle \hat{\Pi}_+ \rangle$ after normalization can be considered as the approximate probability distribution of the arrival time.

5.5.3 Arrival time probability

The operator $\hat{\Pi}_+$ is obtained without specification of the Hamiltonian of the particle and is suitable for free particles and for particles subjected to an external potential as well. In this section we consider the arrival time of the free particle.

The calculation of the “weak arrival time distribution” $W(1, 2)$ involves the average $\langle \hat{\Pi}_+ \rangle$. Therefore, it is useful to have the matrix elements of the operator $\hat{\Pi}_+$. It should be noted that the matrix elements of the operator $\hat{\Pi}_+$ as well as the operator itself are only auxiliary and do not have independent meaning.

In the basis of momentum eigenstates $|p\rangle$, normalized according to the condition $\langle p_1 | p_2 \rangle = 2\pi\hbar\delta(p_1 - p_2)$, the matrix elements of the operator $\hat{\Pi}_+$ are

$$\begin{aligned} \langle p_1 | \hat{\Pi}_+ | p_2 \rangle &= \frac{1}{\Delta t} \langle p_1 | \hat{P}_1 \hat{U}(\Delta t)^\dagger \hat{P}_2 \hat{U}(\Delta t) | p_2 \rangle \\ &= \frac{1}{\Delta t} \int_{-\infty}^X dx_1 \int_X^{\infty} dx_2 e^{-\frac{i}{\hbar} p_1 x_1} \langle x_1 | \hat{U}(\Delta t)^\dagger | x_2 \rangle e^{\frac{i}{\hbar} p_2 x_2 - \frac{i}{\hbar} \frac{p_2^2}{2m} \Delta t}. \end{aligned} \quad (5.105)$$

After performing the integration we obtain

$$\begin{aligned} \langle p_1 | \hat{\Pi}_+ | p_2 \rangle &= \frac{i\hbar}{2\Delta t(p_2 - p_1)} \exp\left(\frac{i}{\hbar}(p_2 - p_1)X\right) \\ &\times \left(e^{\frac{i}{\hbar} \frac{\Delta t}{2m}(p_1^2 - p_2^2)} \operatorname{erfc}\left(-p_1 \sqrt{\frac{i\Delta t}{2\hbar m}}\right) - \operatorname{erfc}\left(-p_2 \sqrt{\frac{i\Delta t}{2\hbar m}}\right) \right), \end{aligned} \quad (5.106)$$

where $\sqrt{i} = \exp(i\pi/4)$. When

$$\frac{1}{\hbar} \frac{\Delta t}{2m} (p_1^2 - p_2^2) \ll 1, \quad p_1 \sqrt{\frac{\Delta t}{2\hbar m}} > 1, \quad p_2 \sqrt{\frac{\Delta t}{2\hbar m}} > 1,$$

the matrix elements of the operator $\hat{\Pi}_+$ are

$$\langle p_1 | \hat{\Pi}_+ | p_2 \rangle \approx \frac{p_1 + p_2}{2m} \exp\left(\frac{i}{\hbar}(p_2 - p_1)X\right). \quad (5.107)$$

This equation coincides with the expression for the matrix elements of the probability current operator.

From Eq. (5.106) we obtain the diagonal matrix elements of the operator $\hat{\Pi}_+$,

$$\langle p | \hat{\Pi}_+ | p \rangle = \frac{p}{2m} \operatorname{erfc}\left(-p \sqrt{\frac{i\Delta t}{2\hbar m}}\right) + \frac{\hbar}{\sqrt{i2\pi\hbar m\Delta t}} e^{-\frac{i}{\hbar} \frac{p^2}{2m} \Delta t}. \quad (5.108)$$

The real part of the quantity $\langle p | \hat{\Pi}_+ | p \rangle$ is shown in Fig. 5.11 and the imaginary part in Fig. 5.12.

Using the asymptotic expressions for the function erfc we obtain from Eq. (5.108) that

$$\lim_{p \rightarrow +\infty} \langle p | \hat{\Pi}_+ | p \rangle \rightarrow \frac{p}{m}$$

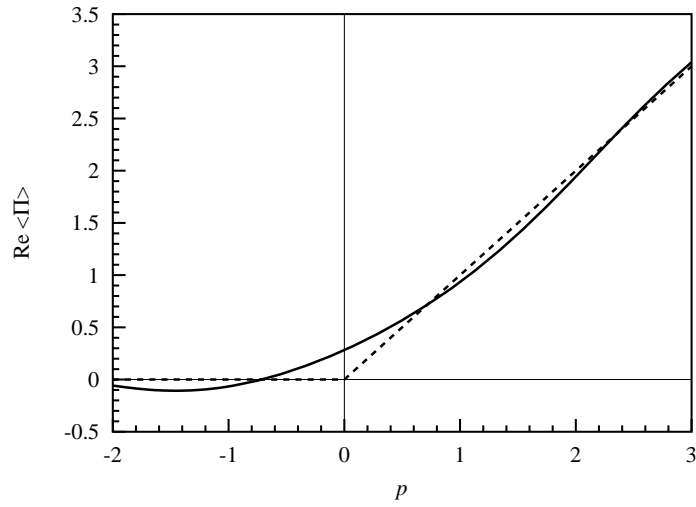


Figure 5.11: The real part of the quantity $\langle p | \hat{\Pi}_+ | p \rangle$, according to Eq. (5.108). The corresponding classical positive probability current is shown with the dashed line. The parameters used are $\hbar = 1$, $m = 1$, and $\Delta t = 1$. In this system of units, the momentum p is dimensionless.

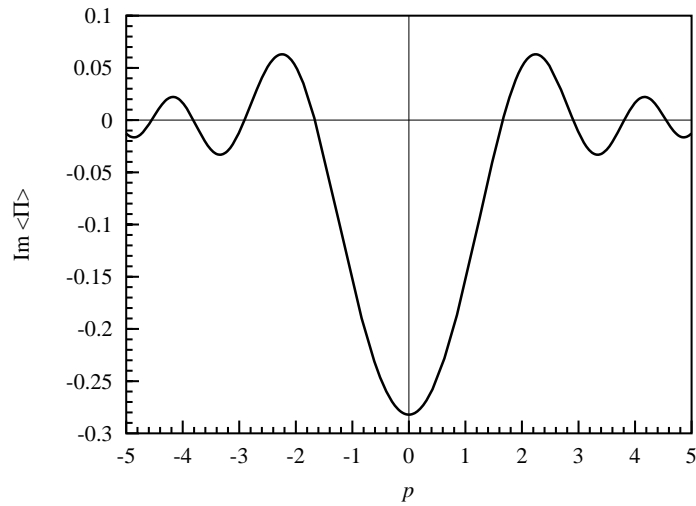


Figure 5.12: The imaginary part of the quantity $\langle p | \hat{\Pi}_+ | p \rangle$. The parameters used are the same as in Fig. 5.11

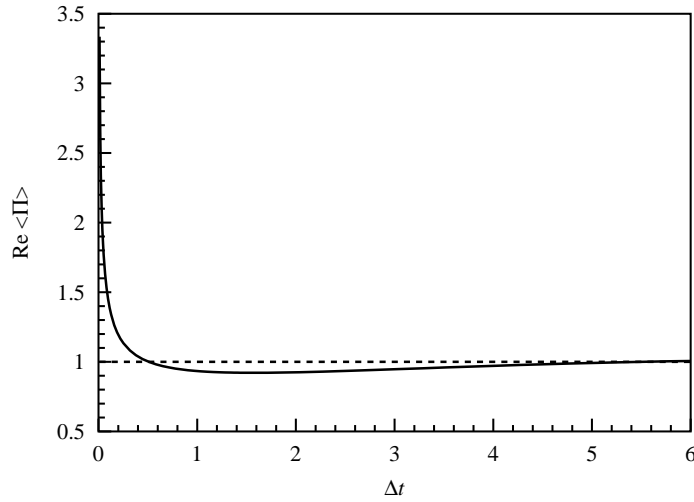


Figure 5.13: The dependence of the quantity $\text{Re}\langle p|\hat{\Pi}_+|p\rangle$ according to Eq. (5.108) on the resolution time Δt . The corresponding classical positive probability current is shown with the dashed line. The parameters used are $\hbar = 1$, $m = 1$, and $p = 1$. In these units, the time Δt is dimensionless.

and $\langle p|\hat{\Pi}_+|p\rangle \rightarrow 0$, when $p \rightarrow -\infty$, i.e., the imaginary part tends to zero and the real part approaches the corresponding classical value as the modulus of the momentum $|p|$ increases. Such behaviour is evident from Figs. 5.11 and 5.12 also.

The asymptotic expressions for function erfc are valid when the argument of the erfc is large, i.e., $|p|\sqrt{\frac{\Delta t}{2\hbar m}} > 1$ or

$$\Delta t > \frac{\hbar}{E_k}. \quad (5.109)$$

Here E_k is the kinetic energy of the particle.

The dependence of the quantity $\text{Re}\langle p|\hat{\Pi}_+|p\rangle$ from Δt is shown in Fig. 5.13. For small Δt the quantity $\langle p|\hat{\Pi}_+|p\rangle$ is proportional to $1/\sqrt{\Delta t}$. Therefore, unlike in classical mechanics, in quantum mechanics Δt cannot be zero. Equation (5.109) imposes the lower bound on the resolution time Δt . It follows that our model does not permit determination of the arrival time with resolution greater than \hbar/E_k . A relation similar to Eq. (5.109) based on measurement models was obtained by Aharonov *et al.* [103]. The time-energy uncertainty relations associated with the time of arrival distribution are also discussed in Refs. [129, 136]

5.6 Summary

The generalization of the theoretical analysis of the time problem in quantum mechanics and weak measurements are presented. The tunneling time problem is a part of this more general problem. The problem of time is solved adapting the weak measurement theory to the measurement of time. In this model expression (5.13) for the duration when the arbitrary observable χ has the certain value is obtained. This result is in agreement with the known results for the dwell time in the tunneling time problem.

Further we consider the problem of the duration when the observable χ has the certain value with condition that the system is in the given final state. Our model of measurement allows us to obtain the expression (5.15) of this duration as well. This expression has many properties of the corresponding classical time. However, such a duration not always has the reasonable meaning. It is possible to obtain the duration the quantity χ has the certain value on condition that the system is in a given final state only when the condition (5.19) is fulfilled. In the opposite case, there is a dependence in the outcome of the measurements on particular detector even in an ideal case and, therefore, it is impossible to obtain the definite value of the duration. When the condition (5.19) is not fulfilled, we introduce two quantities (5.16) and (5.17), characterizing the conditional time. These quantities are useful in the case of tunneling and we suppose that they can be useful also for other problems.

In order to investigate the tunneling time problem, we consider a procedure of time measurement, proposed by Steinberg [104]. This procedure shows clearly the consequences of noncommutativity of the operators and the possibility of determination of the asymptotic time. Our model also reveals the Hartmann and Fletcher effect, i.e., for opaque barriers the effective velocity is very large because the contribution of the barrier region to the time is almost zero. We cannot determine whether this velocity can be larger than c because for this purpose one has to use a relativistic equation (e.g., the Dirac equation).

The definition of density of one sided arrivals is proposed. This definition is extended to quantum mechanics, using the concept of weak measurements by Aharonov *et al.* [94–99]. The proposed procedure is suitable for free particles and for particles subjected to an external potential, as well. It gives not only a mathematical expression for the arrival time probability distribution but also a way of measuring the quantity obtained. However, this procedure gives no unique expression for the arrival time probability distribution.

In analogy with the complex tunneling time, the complex arrival time “probability distribution” is introduced (Eq. (5.102)). It is shown that the proposed approach imposes an inherent limitation, Eq. (5.109), on the resolution time of the arrival time determination.

6 Summary of the results and conclusions

1. A simple model of the measurement without taking into account the interaction of the detector with the environment is considered. Using this model of the measurement the general equation for the probability of the jump during the measurement is derived. The behavior of the system under the repeated measurements depends on the strength of the interaction with the measuring device and on the properties of the system.
2. When the the strength of the interaction with the measuring device is sufficiently large, the frequent measurements of the system slow down the evolution. However, the evolution cannot be fully stopped.
3. Measurements can cause inhibition or acceleration of the evolution. The decay rate may be increased by the measurements and the quantum anti-Zeno effect can be obtained.
4. General expression for the jump probability during the measurement, first obtained by Kofman and Kurizki [26], is derived. The main assumptions, used in the derivation, are the assumptions that the quantum measurement is non-demolition measurement and the Markovian approximation for the quantum dynamics of the detector is valid. We have shown that this expression is also suitable for the description of the pulsed measurements, when there are intervals of the measurement-free evolution between the successive measurements.
5. A model of the measurement in which the detector is modeled as a harmonic oscillator, initially being at the thermal equilibrium, is investigated. The Lindblad-type master equation for the detectors density matrix is solved analytically. An equation for the probability of the jump between measured system's states during the measurement is obtained. From the used model it follows that the increase of the detector's temperature leads to the enhancement of the quantum Zeno or quantum anti-Zeno effects.
6. In another model the detector is a two level system interacting with the environment. The influence of the environment is taken into account using quantum trajectory method. The quantum trajectories produced by stochastic simulations show the probabilistic behavior exhibiting the collapse of the wave-packet in the measured system, although the quantum jumps were performed only in the detector. Both quantum Zeno and anti-Zeno effects were demonstrated for the measured two level system and for the decaying system. A good agreement of the numerical results

with the analytical estimates of the decay rates of the measured system shows that the particular model of the detector is not important, since the decay rates mostly depend only on one parameter, i.e., the duration of the measurement.

7. Weak measurement theory by Aharonov *et al.* is used to investigate the measurement of time in quantum mechanics. An expression for the duration when the arbitrary observable χ has the certain value is obtained. This result is in agreement with the known results for the dwell time in the tunneling time problem. The expression for the duration when the observable χ has the certain value with condition that the system is found in a given final state is introduced. This expression has many properties of the corresponding classical time. However, it is shown that this duration has definite value only when commutativity condition (5.19) is fulfilled. In the opposite case two characteristic durations can be introduced that can be combined into one complex quantity.
8. Particular case of this duration is tunneling time. Weak measurement procedure shows clearly the consequences of noncommutativity of the operators — we cannot simultaneously have the information about the tunneling and location of the particle. Our model also reveals the Hartmann and Fletcher effect, i.e., for opaque barriers the contribution of the barrier region to the time is almost zero.
9. The definition of density of one sided arrivals is proposed. This definition is extended to quantum mechanics, using the concept of weak measurements. The proposed procedure is suitable for free particles and for particles subjected to an external potential, as well. However, this procedure gives no unique expression for the arrival time probability distribution. In analogy with the complex tunneling time, the complex arrival time “probability distribution” is introduced. It is shown that the proposed approach imposes an inherent limitation on the resolution of the arrival time.

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Santrauka

Disertacija skirta paprastų matavimo modelių kvantinėje mechanikoje nagrinėjimui ir tu modelių taikymui kvantinių Zenono ir anti-Zenono efektų aprašymui bei laiko apibrėžimo problemoms kvantinėje mechanikoje spręsti.

Kvantiniai Zenono ir anti-Zenono efektai nagrinėjami naudojant skirtingus matavimo modelius. Modeliuose atsižvelgiama į baigtinę matavimų trukmę bei tikslumą.

Iš pradžių nagrinėjamas paprastas matavimo modelis, neįskaitantis detektoriaus sąveikos su aplinka. Naudojantis šiuo modeliu išvesta bendra formulė šuolio matavimo metu tikimybei. Dažnai matuojamos sistemos elgesys priklauso nuo sąveikos su matavimo įrenginiu stiprumo ir nuo pačios sistemos sąvybių. Dažni matavimai gali sukelti sistemos evoliucijos pagreitėjimą arba sulėtėjimą.

Toliau kvantiniai Zenono ir anti-Zenono efektai analizuojami nekonkretizuojant matavimo modelio bei išvesta bendra formulė šuolio matavimo metu tikimybei. Pagrindinės išvedimui naudotos prielaidos: matavimas yra netrikdantis ir detektoriaus dinamikai yra tinkamas Markovo artinys.

Išnagrinėtas matavimo modelis, kuriame detektorius yra harmoninis osciliatorius, iš pradžių esantis šiluminėje pusiausvyroje su aplinka. Analiziškai išspręsta Lindblad'o tipo lygtis detektoriaus tankio matricai. Iš naudoto modelio seka, kad detektoriaus temperatūros padidėjimas sustiprina kvantinį Zenono ar anti-Zenono efektą.

Kitame iš nagrinėtų modelių detektorius yra dviejų lygmenų sistema, sąveikaujanti su aplinka. Į aplinkos įtaką atsižvelgiama naudojant kvantinių trajektorijų metodą. Tiek kvantinis Zenono, tiek anti-Zenono efektai yra pademonstruoti matuojamai dviejų lygmenų sistemai bei skylančiai sistemai. Gautas geras skaitmeninių rezultatų sutapimas su analiziniu matuojamos sistemos skilimo spartos įvertinimu.

Silpni matavimai, pasiūlyti Ahronov'o, Albert'o ir Vaidman'o, panaudoti laiko matavimo kvantinėje mechanikoje tyrimui. Įvesta išraiška trukmei per kurią dydis turi nurodytą reikšmę su sąlyga, kad sistema yra rasta nurodytoje galinėje būsenoje. Ši išraiška turi daug atitinkamos klasikinės trukmės sąvybių. Tačiau ji turi apibrėžtą reikšmę tik kai (5.19) komutavimo sąlyga yra patenkinta. Priešingu atveju gali būti įvestos dvi charakteringos trukmės.

Atskiras tokios trukmės atvejis yra tuneliavimo trukmė. Silpnų matavimų procedūra parodo operatorių nekomutavimo pasekmes: negalime vienu metu turėti informacijos apie tai, ar dalelė tuneliavo, ir apie dalelės padėtį. Nagrinėtas modelis taip pat demonstruoja Hartmann'o ir Fletcher'io efektą: platiems barjerams barjero srities indėlis į tuneliavimo trukmę yra beveik lygus nuliui.

Pasiūlytas atvykimo laiko tikimybės tankio apibrėžimas. Šis apibrėžimas išplėstas į kvantinę mechaniką, naudojantis silpnais matavimais. Tačiau ši procedūra neduoda vienareikšmio atvykimo laiko tikimybės pasiskirstymo. Analogiškai kompleksinei tuneliavimo trukmei įvedamas kompleksinis atvykimo laiko pasiskirstymas. Parodyta, kad pasiūlytas metodas turi apribojimą atvykimo laiko nustatymo tikslumui.