Phase-space curvature in spin-orbit coupled ultracold atomic systems

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Abstract

The Berry phase, as well as Berry curvatures in real and momentum spaces, has been thoroughly discussed in the literature in various contexts [1]. Here we approach the phase-space Berry curvature with applications in ultracold-atom systems in mind. We consider ultracold atom systems with artificially engineered spin-orbit coupling [2], which have recently attracted considerable attention. We derive quantum-mechanical Heisenberg equations of motion where position-space, momentum-space, and phase-space Berry curvatures show up without relying on the semiclassical approximation [3]. Subsequently, we perform the semiclassical approximation and obtain the semiclassical equations of motion. We show that in the semiclassical regime the effective mass of the equal Rashba-Dresselhaus spin-orbit-coupled system can be viewed as a direct effect of the phase-space Berry curvature.

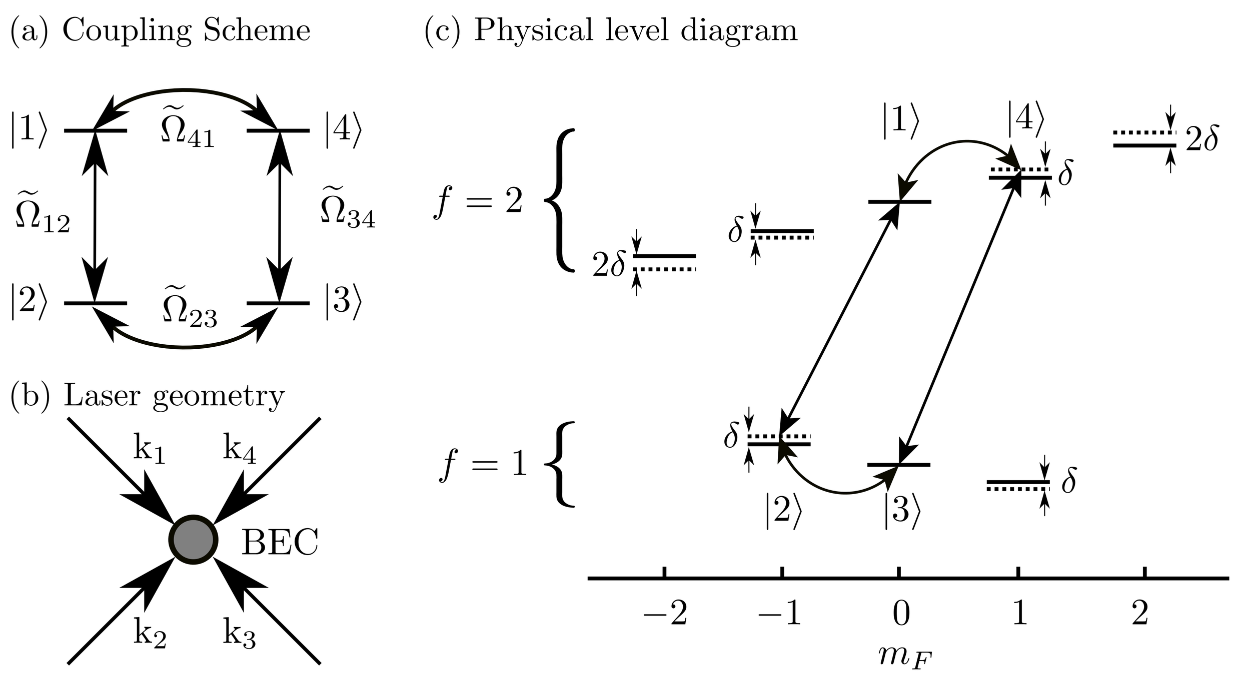
[1] N. Goldman, G. Juzeliūnas, P. Öhberg, and I. B. Spielman, Rep. Prog. Phys. **77**, 126401 (2014).

[2] J. Dalibard, F. Gerbier, G. Juzeliūnas, and P. Öhberg, Rev. Mod. Phys. **83**, 1523 (2011).

[3] J. Armaitis, J. Ruseckas, and E. Anisimovas, Phys. Rev. A **95**, 043616 (2017).

Adiabatic approximation and spin-orbit coupling

Postion-dependent spin-orbit coupling for ultracold atoms



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Hamiltonian with position-dependent spin-orbit coupling

$$H = \frac{\hbar^2}{2m}(\boldsymbol{k} - \boldsymbol{A}(\boldsymbol{r}))^2 + V(\boldsymbol{r}) \equiv \frac{\hbar^2}{2m}\boldsymbol{k}^2 + \vec{B}\cdot\vec{\sigma} + W(\boldsymbol{r})I\,,$$

where $\boldsymbol{A}$ and V are 2×2 matrices:

$$V(\boldsymbol{r}) = \vec{v}(\boldsymbol{r})\cdot\vec{\sigma} + v_0(\boldsymbol{r})I\,, \qquad A_j(\boldsymbol{r}) = \vec{a}_j(\boldsymbol{r})\cdot\vec{\sigma}$$

Adiabatic approximation

Unitary operator U diagonalizes the term $\vec{B}\cdot\vec{\sigma}$. Adiabatic approximation: the wave function remains in either the lower or upper dispersion branch with respect to the position- and momentum-dependent effective magnetic field $\vec{B}$; $\Psi = \psi\mathcal{P}_\pm$, where $\mathcal{P}_\pm$ are eigenstates of σ_z. In the effective Hamiltonian

$$H_{\text{eff}} = \mathcal{P}_\pm^\dagger U^\dagger H U \mathcal{P}_\pm$$

the covariant operators appear:

$$\boldsymbol{r}_c = \mathcal{P}_\pm^\dagger U^\dagger \boldsymbol{r} U\mathcal{P}_\pm\,, \qquad \boldsymbol{k}_c = \mathcal{P}_\pm^\dagger U^\dagger \boldsymbol{k} U\mathcal{P}_\pm$$

Covariant operators can be written as

$$\boldsymbol{r}_c = \boldsymbol{r} - \boldsymbol{\mathcal{A}}^{(k)}\,, \qquad \boldsymbol{k}_c = \boldsymbol{k} - \boldsymbol{\mathcal{A}}^{(r)}\,,$$

where the operators

$$\boxed{\boldsymbol{\mathcal{A}}^{(k)} = -\mathcal{P}_\pm^\dagger U^\dagger[\boldsymbol{r}, U]\mathcal{P}_\pm\,, \qquad \boldsymbol{\mathcal{A}}^{(r)} = -\mathcal{P}_\pm^\dagger U^\dagger[\boldsymbol{k}, U]\mathcal{P}_\pm}$$

correspond to Berry connections. Commutators:

$$[(r_c)_j, (r_c)_l] = \mathrm{i}\Theta_{jl}^{(k,k)}\,, \qquad [(k_c)_j, (k_c)_l] = \mathrm{i}\Theta_{jl}^{(r,r)}\,, \qquad [(r_c)_j, (k_c)_l] = \mathrm{i}\delta_{j,l} + \mathrm{i}\Theta_{jl}^{(k,r)}$$

where various Berry curvatures are given by

$$\boxed{\begin{aligned} \Theta_{jl}^{(k,k)} &= \mathrm{i}[r_j, \mathcal{A}_l^{(k)}] - \mathrm{i}[r_l, \mathcal{A}_j^{(k)}]\,, & \Theta_{jl}^{(r,r)} &= \mathrm{i}[k_j, \mathcal{A}_l^{(r)}] - \mathrm{i}[k_l, \mathcal{A}_j^{(r)}] \\ \Theta_{jl}^{(k,r)} &= \mathrm{i}[r_j, \mathcal{A}_l^{(r)}] - \mathrm{i}[k_l, \mathcal{A}_j^{(k)}]\,, & \Theta_{jl}^{(r,k)} &= \mathrm{i}[k_j, \mathcal{A}_l^{(k)}] - \mathrm{i}[r_l, \mathcal{A}_j^{(r)}] \end{aligned}}$$

The effective Hamiltonian can be written as

$$H_{\text{eff}} = \frac{\hbar^2}{2m}\boldsymbol{k}_c^2 + W(\boldsymbol{r}_c) + \mathcal{V}$$

where

$$\mathcal{V} = \mathcal{P}_\pm^\dagger U^\dagger \vec{B}\cdot\vec{\sigma} U\mathcal{P}_\pm + \mathcal{V}^{(r)} + \mathcal{V}^{(k)}$$

with

$$\mathcal{V}^{(r)} = \frac{\hbar^2}{2m}\mathcal{P}_\pm U^\dagger[\boldsymbol{k}, U]\mathcal{P}_\mp\cdot\mathcal{P}_\mp U^\dagger[\boldsymbol{k}, U]\mathcal{P}_\pm\,, \qquad \mathcal{V}^{(k)} = \sum_{j,l} w_{jl}^{(2)}\mathcal{P}_\pm U^\dagger[r_j, U]\mathcal{P}_\mp\mathcal{P}_\mp U^\dagger[r_l, U]\mathcal{P}_\pm$$

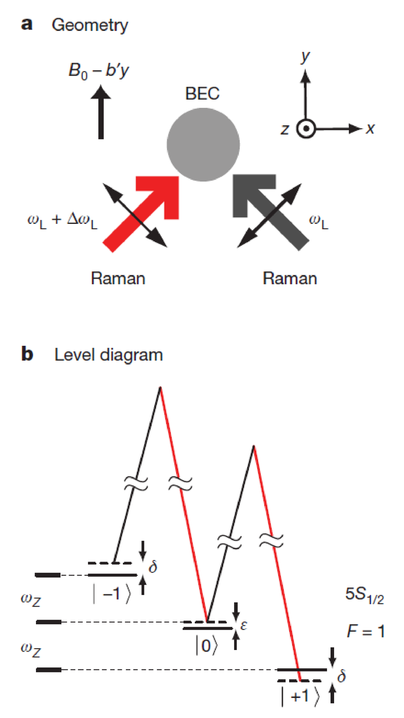
Here we assumed, that the potential $W(\boldsymbol{r})$ is at most quadratic.
Heisenberg equations for the covariant operators

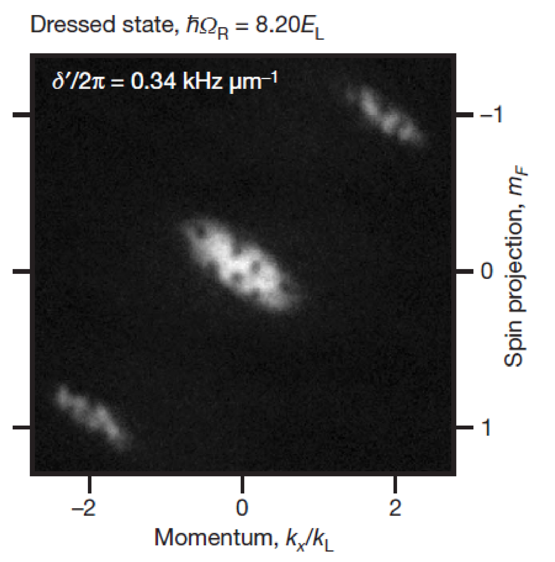
$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{k}_c = \frac{1}{\mathrm{i}\hbar}[\boldsymbol{k}_c, H_{\text{eff}}]\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_c = \frac{1}{\mathrm{i}\hbar}[\boldsymbol{r}_c, H_{\text{eff}}]$$

contain Berry curvature terms:

$$\boxed{\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{k}_c &= -\frac{1}{\hbar}\boldsymbol{\nabla}W(\boldsymbol{r}_c) + \frac{1}{\mathrm{i}\hbar}[\boldsymbol{k}_c, \mathcal{V}] + \frac{\hbar}{2m}\sum_{j,l}\boldsymbol{e}_j\left\{\Theta_{jl}^{(r,r)}, (k_c)_l\right\} + \frac{1}{2\hbar}\sum_{j,l}\boldsymbol{e}_j\left\{\Theta_{jl}^{(r,k)}, \nabla_l W(\boldsymbol{r}_c)\right\} \\ \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_c &= \frac{\hbar}{m}\boldsymbol{k}_c + \frac{1}{\mathrm{i}\hbar}[\boldsymbol{r}_c, \mathcal{V}] + \frac{\hbar}{2m}\sum_{j,l}\boldsymbol{e}_j\left\{\Theta_{jl}^{(k,r)}, (k_c)_l\right\} + \frac{1}{2\hbar}\sum_{j,l}\boldsymbol{e}_j\left\{\Theta_{jl}^{(k,k)}, \nabla_l W(\boldsymbol{r}_c)\right\} \end{aligned}}$$

Particular case: synthetic magnetic field





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Semiclassical approximation

We neglect the commutator between position and momentum. Eigenvectors $\chi_\pm(\boldsymbol{r}, \boldsymbol{k})$ of the matrix $\vec{B}\cdot\vec{\sigma}$ prametrically depend on the numbers $\boldsymbol{r}$ and $\boldsymbol{k}$. Berry connections:

$$\boldsymbol{\mathcal{A}}^{(k)} = \mathrm{i}\chi_\pm^\dagger\boldsymbol{\nabla}^{(r)}\chi_\pm\,, \qquad \boldsymbol{\mathcal{A}}^{(r)} = -\mathrm{i}\chi_\pm^\dagger\boldsymbol{\nabla}^{(k)}\chi_\pm$$

Berry curvatures:

$$\begin{aligned} \Theta_{jl}^{(k,k)} &= -\nabla_j^{(k)}\mathcal{A}_l^{(k)} + \nabla_l^{(k)}\mathcal{A}_j^{(k)}\,, & \Theta_{jl}^{(r,r)} &= \nabla_j^{(r)}\mathcal{A}_l^{(r)} - \nabla_l^{(r)}\mathcal{A}_j^{(r)} \\ \Theta_{jl}^{(k,r)} &= -\nabla_j^{(k)}\mathcal{A}_l^{(r)} - \nabla_l^{(r)}\mathcal{A}_j^{(k)}\,, & \Theta_{jl}^{(r,k)} &= \nabla_j^{(r)}\mathcal{A}_l^{(k)} + \nabla_l^{(k)}\mathcal{A}_j^{(r)} \end{aligned}$$

Scalar potentials:

$$\mathcal{V}^{(r)} = -\frac{\hbar^2}{2m}\chi_\pm^\dagger\boldsymbol{\nabla}^{(r)}\chi_\mp\cdot\chi_\mp^\dagger\boldsymbol{\nabla}^{(r)}\chi_\pm\,, \qquad \mathcal{V}^{(k)} = -\sum_{j,l} w_{jl}^{(2)}\chi_\pm^\dagger\nabla_j^{(k)}\chi_\mp\chi_\mp^\dagger\nabla_l^{(k)}\chi_\pm$$

Equations of motion

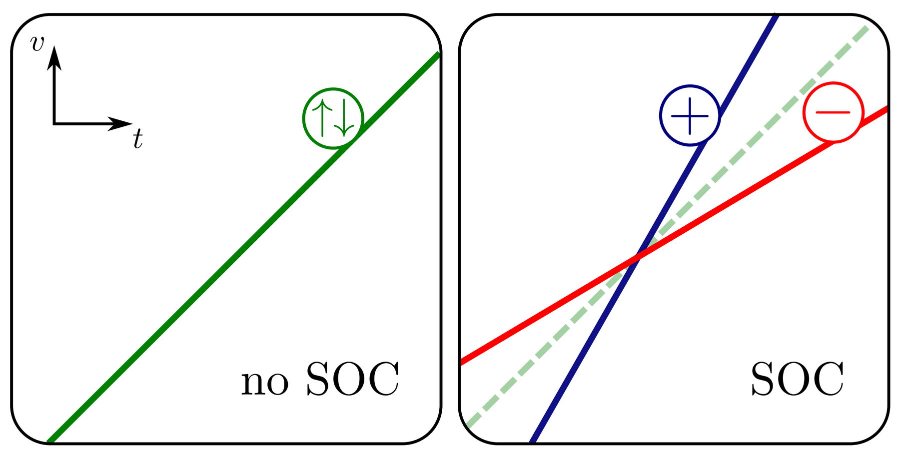
$$\boxed{\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{k}_c &= -\frac{1}{\hbar}\boldsymbol{\nabla}W(\boldsymbol{r}_c) - \frac{1}{\hbar}\boldsymbol{\nabla}^{(r)}\mathcal{V} + \frac{\hbar}{m}\sum_{j,l}\boldsymbol{e}_j\Theta_{jl}^{(r,r)}(k_c)_l + \frac{1}{\hbar}\sum_{j,l}\boldsymbol{e}_j\Theta_{jl}^{(r,k)}\nabla_l W(\boldsymbol{r}_c) \\ \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_c &= \frac{\hbar}{m}\boldsymbol{k}_c + \frac{1}{\hbar}\boldsymbol{\nabla}^{(k)}\mathcal{V} + \frac{\hbar}{m}\sum_{j,l}\boldsymbol{e}_j\Theta_{jl}^{(k,r)}(k_c)_l + \frac{1}{\hbar}\sum_{j,l}\boldsymbol{e}_j\Theta_{jl}^{(k,k)}\nabla_l W(\boldsymbol{r}_c) \end{aligned}}$$

Spin-orbit coupling in one dimension

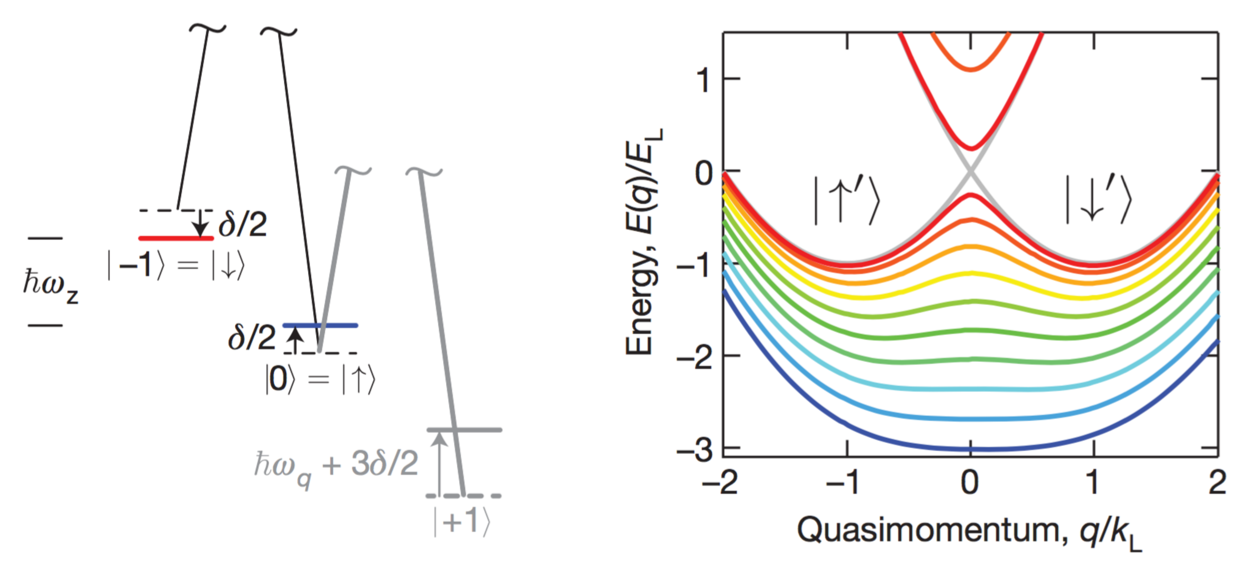
Let us consider the system with the Hamiltonian

$$H = H_0 - Fx\,, \qquad H_0 = \frac{\hbar^2}{2m}(k - a\sigma_3)^2 + \frac{\hbar\Omega}{2}[\cos(x/\lambda)\sigma_1 + \sin(x/\lambda)\sigma_2]$$

Response to a force: effective mass



Hamiltonian H_0 has been realized experimentally



Y.-J. Lin, K. Jiménez-García and I. B. Spielman, Nature **471**, 83–86 (2011).

Exact solution: effective mass

$$\frac{m}{m_\pm^*} = 1 \pm \frac{k_0^2}{\kappa a} \approx 1 \pm \frac{1}{\kappa}\left(a + \frac{1}{\lambda}\right)\,; \qquad k_0 = a + \frac{1}{2\lambda}\,, \qquad \kappa = \frac{\Omega}{2a}\frac{m}{\hbar}$$

Semiclassical dynamics follows the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{k}_c = \frac{F}{\hbar}(1 - \Theta^{(r,k)})\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}_c = \frac{\hbar}{m}\boldsymbol{k}_c(1 + \Theta^{(k,r)}) + \frac{1}{\hbar}\nabla^{(k)}\mathcal{V}$$

In the limit $1/\lambda a \ll 1$ and $|k| \ll \kappa$,

$$\Theta^{(r,k)} = -\Theta^{(k,r)} \approx \mp\frac{1}{2\lambda\kappa}$$

Closed equation for the center of mass motion

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{x}_c = \frac{F}{m}\left(1 \pm \frac{1}{\kappa}\left(a + \frac{1}{\lambda}\right)\right)$$

The effective mass in semiclassical approximation is correctly captured by the phase-space Berry curvature.

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