Nonextensive statistics with application to financial processes based on nonlinear stochastic differential equations

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Abstract

Starting from the multiplicative point process and nonlinear stochastic models of $1/f$ noise and power-law distributions [1, 2] we present nonlinear stochastic differential equations generating processes with the q-exponential and q-Gaussian distributions of the observables with the long-range power-law autocorrelations and $1/f^{\beta}$ noise [3]. Further we analyze properties of solutions of these equations in relation with the superstatistical approach [4] and relevance of the generalized and adapted equations for modeling of the financial processes [5].

Stochastic model with q-Gaussian PDF and long memory

We have proposed **nonlinear** SDE [1, 2]

$$\mathrm{d}x = \sigma^2\left(\eta - \frac{\lambda}{2}\right)x^{2\eta-1}\mathrm{d}t + \sigma x^{\eta}\mathrm{d}W \tag{1}$$

Due to requirement of the stationarity of the process, the stochastic equation (1) should be analyzed together with the appropriate restrictions of the diffusion in some finite interval $x_{\min} \leqslant x \leqslant x_{\max}$ [1, 2]. Solutions of SDE (1) exibit power-law probability distribution function, $P(x) \sim x^{-\lambda}$, and power spectral density, $S(f) \sim 1/f^{\beta}$, in wide range of frequencies. Here

$$\beta = 1 - \frac{\lambda-3}{2\eta-2}, \qquad \frac{1}{2} < \beta < 2\,. \tag{2}$$

Power spectral density is determined mainly by the power-law behavior of the coefficients of SDE (1) at big values of $x \gg x_{\min}$. Changing the coefficients at small x, the spectrum retains power-law behavior. Therefore, we propose [3, 5] the following **modification** of Eq. (1)

$$\mathrm{d}x = \sigma^2\left(\eta - \frac{\lambda}{2}\right)(x_0^2 + x^2)^{\eta-1}x\mathrm{d}t + \sigma(x_0^2+x^2)^{\eta/2}\mathrm{d}W. \tag{3}$$

The associated Fokker-Planck equation gives q-Gaussian PDF,

$$P(x) = \frac{\Gamma(\lambda/2)}{\sqrt{\pi}x_0\Gamma((\lambda-1)/2)}\left(\frac{x_0^2}{x_0^2+x^2}\right)^{\lambda/2} \equiv A_q \exp_q\left(-\lambda\frac{x^2}{(3-q)\sigma_q^2}\right), \tag{4}$$

$$q = 1 + 2/\lambda. \tag{5}$$

Here $\exp_q(\cdot)$ is q-exponential defined as $\exp_q(x) \equiv (1+(1-q)x)^{1/(1-q)}$.

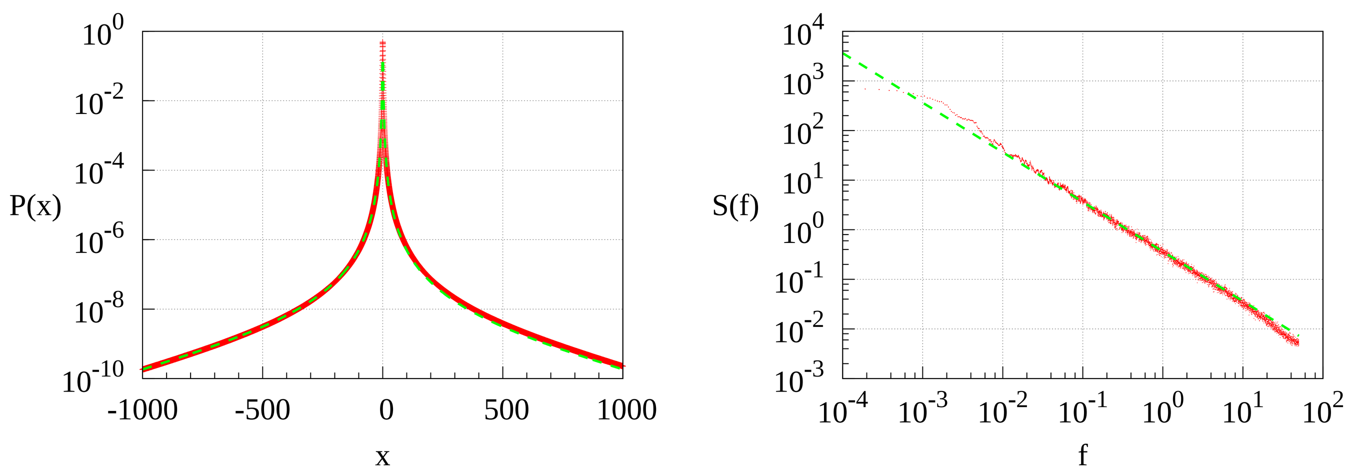

FIGURE 1: PDF, $P(x)$, of solution of Eq. (3) with parameters $x_0 = 1$, $\sigma = 1$, $\lambda = 3$ and $\eta = 5/2$ coinciding with the analytical Eq. (4) and spectrum, $S(f)$, in comparison with the line representing $1/f$ spectrum.

Equation with two power-law exponents

In order to get spectrum with two power-law exponents we propose SDE [5]

$$\mathrm{d}\tilde{x} = \left(\eta - \frac{\lambda_0}{2} - \left(\frac{\tilde{x}}{\tilde{x}_{\max}}\right)^2\right)\frac{(1+\tilde{x}^2)^{\eta-1}}{((1+\tilde{x}^2)^{1/2}\epsilon+1)^2}\tilde{x}\mathrm{d}\tilde{t} + \frac{(1+\tilde{x}^2)^{\eta/2}}{(1+\tilde{x}^2)^{1/2}\epsilon+1}\mathrm{d}\tilde{W}. \tag{6}$$

Here $\tilde{x} = x/x_0$ and $\tilde{t} = t\sigma^2 x_0^{2(\eta-1)}$ are scaled variables. We demonstrate an example solutions of equation (6) in figure 2.

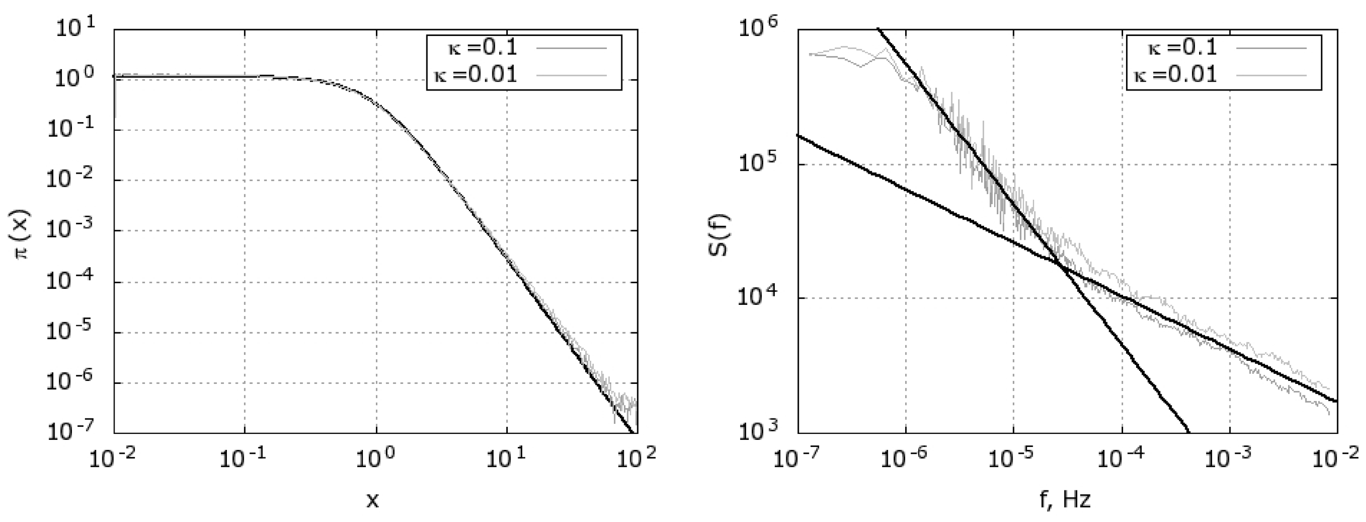

FIGURE 2: Model calculated from Eq. (6) with the parameters $\eta = 5/2$, $\lambda_0 = 4.0$, $\tilde{x}_{\max} = 10^5$ and $\epsilon = 0.01$ PDF, $P(x)$, (continuum line) in comparison with empirical histogram of one minute returns of ABT stocks traded on NYSE (dots) and power spectrum, $S(x)$, of returns.

Nonlinear stochastic model of return

In the model proposed we assume that the empirical return r_t can be written as instantaneous q-Gaussian fluctuations ξ with a slowly diffusing parameter r_0 and constant $\lambda = 5$.

$$r_t = \xi\{r_0, \lambda\}. \tag{7}$$

The parameter r_0 serves as a measure of instantaneous volatility of return fluctuations. We do propose to model the measure of volatility r_0 by the scaled continuous stochastic variable x, having a meaning of average return per unit time interval. By the empirical analyses of high frequency trading data on NYSE we introduced relation

$$r_0(t,\tau) = 1 + \frac{\bar{r}_0}{\tau_s}\left|\int_{t_s}^{t_s+\tau_s} x(s)\mathrm{d}s\right|, \tag{8}$$

where $\bar{r}_0$ is an empirical parameter and the average return per unit time interval $x(t_s)$ can be modeled by the nonlinear SDE (6).
Empirically defined parameters: η - exponent of multiplicativity, λ_0 - power law exponent of x long range PDF, ϵ - parameter dividing diffusion into two areas: stationary and excited one, and $x_{\max}$ - the upper limit of diffusion.

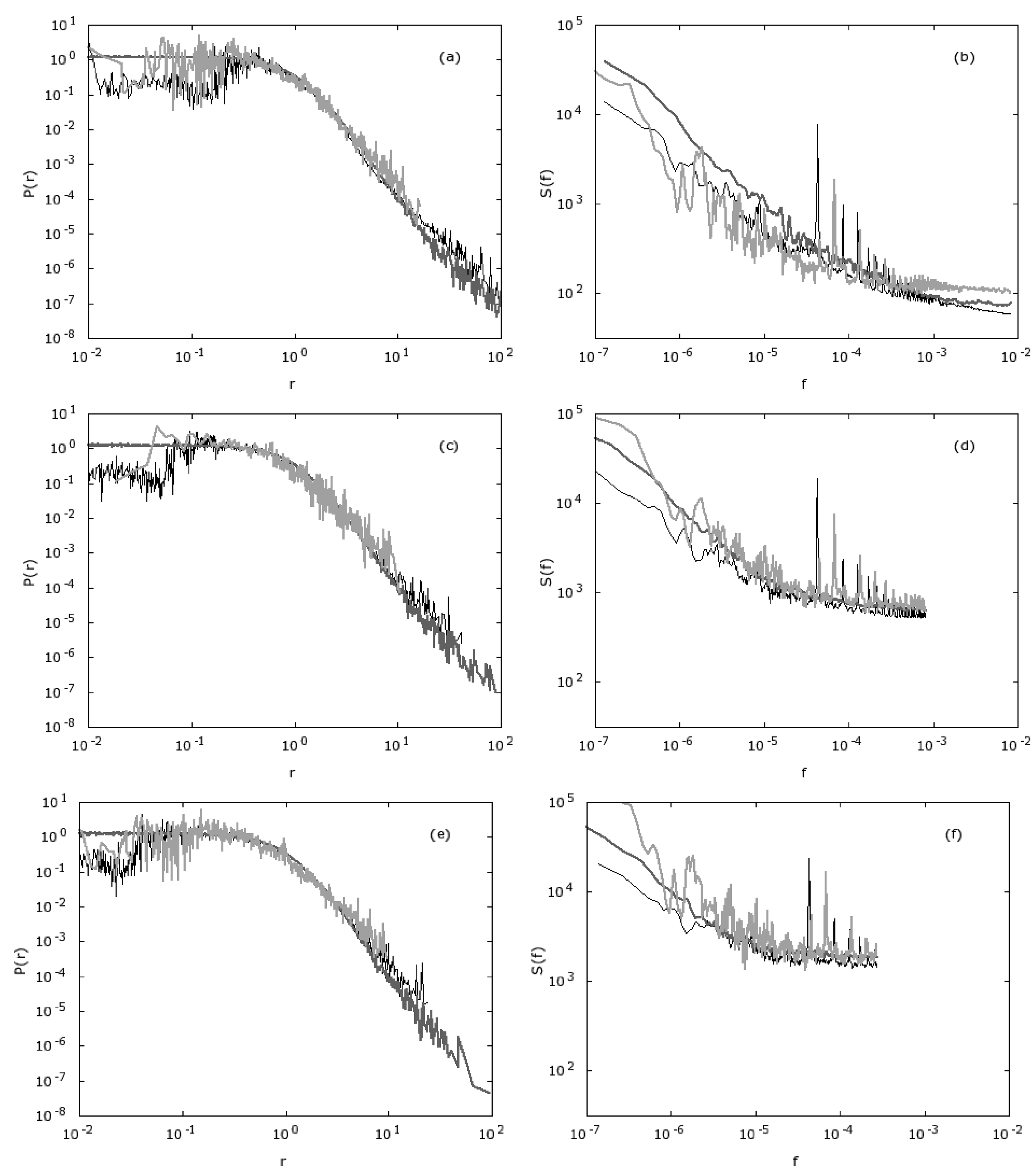

FIGURE 3: Comparison of empirical statistics of absolute returns traded on the NYSE (black thin lines) and VSE (light gray lines) with model statistics, (gray lines). Model parameters are as follows: $\lambda = 5$; $\sigma_t^2 = 1/3 \cdot 10^{-6}\,\mathrm{s}^{-1}$; $\lambda_0 = 3.6$; $\epsilon = 0.017$; $\eta = 2.5$; $\bar{r}_0 = 0.4$; $x_{\max} = 1000$. PDF of normalized absolute returns is given on (a),(c),(e) and PSD on (b),(d),(f). (a) and (b) represents results with $\tau = 60\,\mathrm{s}$; (c) and (d) $\tau = 600\,\mathrm{s}$; (e) and (f) $\tau = 1800\,\mathrm{s}$. Empirical data from NYSE is averaged over 24 stocks and empirical data from VSE is averaged over 4 stocks.

Conclusions

We proposed a double stochastic process driven by the nonlinear scaled SDE reproducing the main statistical properties of the absolute return, observed in the financial markets. Seven parameters of the model enable us to adjust it to the power law statistics of various stocks including long range behavior. All parameters introduced are recoverable from the empirical data and are responsible for the specific statistical features of real markets. Seeking to discover the universal nature of return statistics we analyse and compare extremely different markets in New York and Vilnius and adjust the model parameters to match statistics of both markets.

Acknowledgements

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