# $1/f$ noise and $q$-Gaussian distribution from nonlinear stochastic differential equations

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## Abstract

There exist a number of systems, involving long-range interactions, long-range memory, and anomalous diffusion, that possess anomalous properties in view of traditional Boltzmann-Gibbs statistical mechanics. Nonextensive statistical mechanics represents a consistent theoretical background for the investigation of some properties, like fractality, multifractality, self-similarity, long-range dependencies of such complex systems. We present nonlinear stochastic differential equations (SDEs) yielding $q$-Gaussian distribution of signal intensity, featured in nonextensive statistical mechanics, as well as the long-range power-law autocorrelations and $1/f^\beta$ power spectral density ($1/f$ noise). The Tsallis $q$-distributions also may be obtained in the superstatistical framework as a superposition of different local dynamics at different time intervals. We analyze relevance of the generalized and adapted equations for modeling the financial processes. We model the inter-trade durations, the trading activity and the normalized return using the superstatistical approaches with the exponential and normal distributions of the local signals driven by the nonlinear stochastic process.

1. J. Ruseckas and B. Kaulakys, Phys. Rev. E **84**, 051125 (2011).
2. J. Ruseckas, V. Gontis and B. Kaulakys, Advances in Complex Systems **15** Suppl. 1 1250073 (2012).

## Nonlinear stochastic differential equations gnenerating signals with $1/f^\beta$ noise

Proposed Itô SDEs

$$\boxed{dx=\sigma^2\left(\eta-\frac{1}{2}\lambda\right)x^{2\eta-1}dt+\sigma x^\eta dW}$$

The diffusion of stochastic variable $x$ should be restricted.

Steady-state probability density function (PDF):

$$P(x)\sim x^{-\lambda}$$

Power spectral density (PSD):

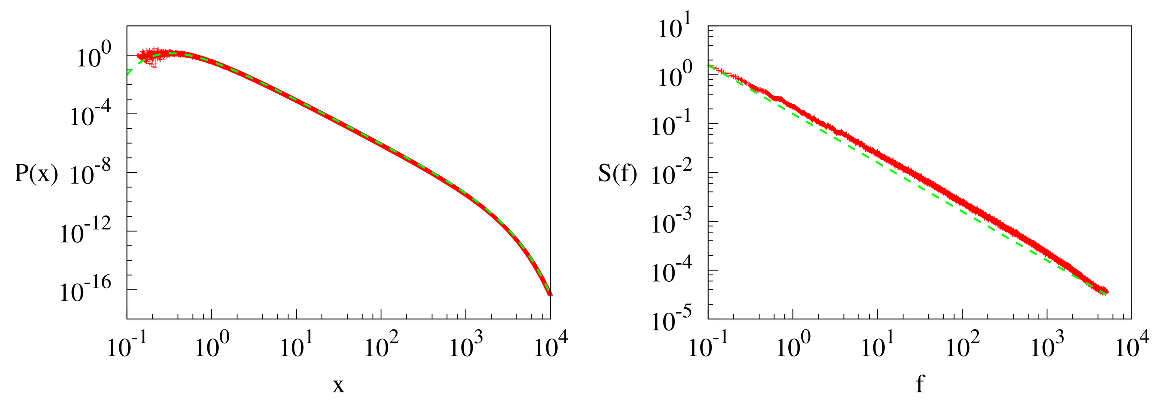
$$S(f)\sim\frac{1}{f^\beta}\,,\qquad \beta=1+\frac{\lambda-3}{2(\eta-1)}$$

SDE

$$dx=\sigma^2\left[\eta-\frac{1}{2}\lambda+\frac{m}{2}\left(\frac{x_{\min}^m}{x^m}-\frac{x^m}{x_{\max}^m}\right)\right]x^{2\eta-1}dt+\sigma x^\eta dW$$

yelds steady state PDF

$$P(x)\sim\frac{1}{x^\lambda}\exp\left\{-\left(\frac{x_{\min}}{x}\right)^m-\left(\frac{x}{x_{\max}}\right)^m\right\}$$



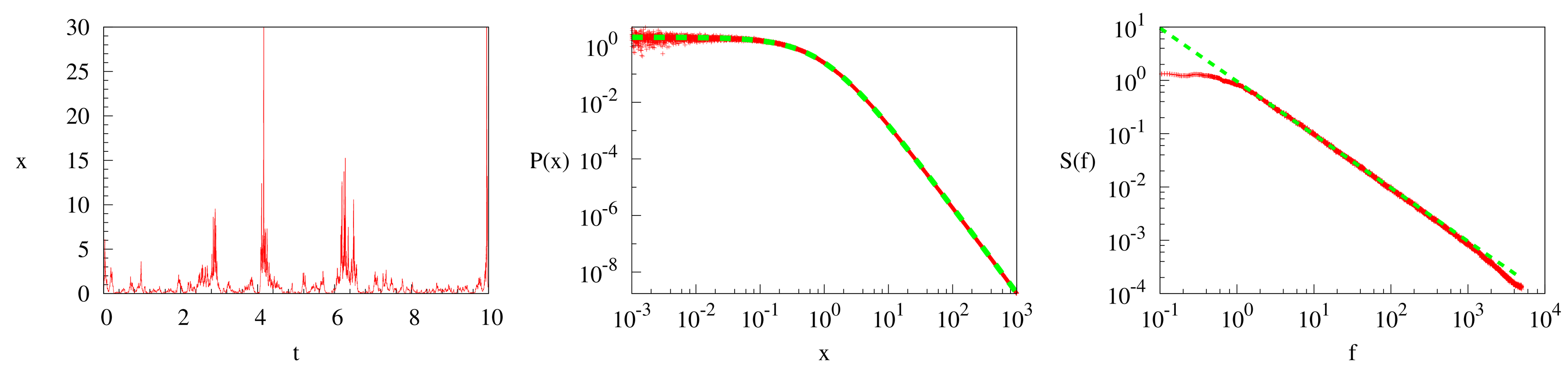

## $q$-exponential distribution

SDE

$$dx=\sigma^2\left(\eta-\frac{1}{2}\lambda\right)(x+x_0)^{2\eta-1}dt+\sigma(x+x_0)^\eta dW$$

with reflective boundary at $x=0$. Steady-state PDF:

$$P(x)=\frac{\lambda-1}{x_0}\left(\frac{x_0}{x+x_0}\right)^\lambda=\frac{\lambda-1}{x_0}\exp_q(-\lambda x/x_0)\,,\qquad q=1+1/\lambda$$



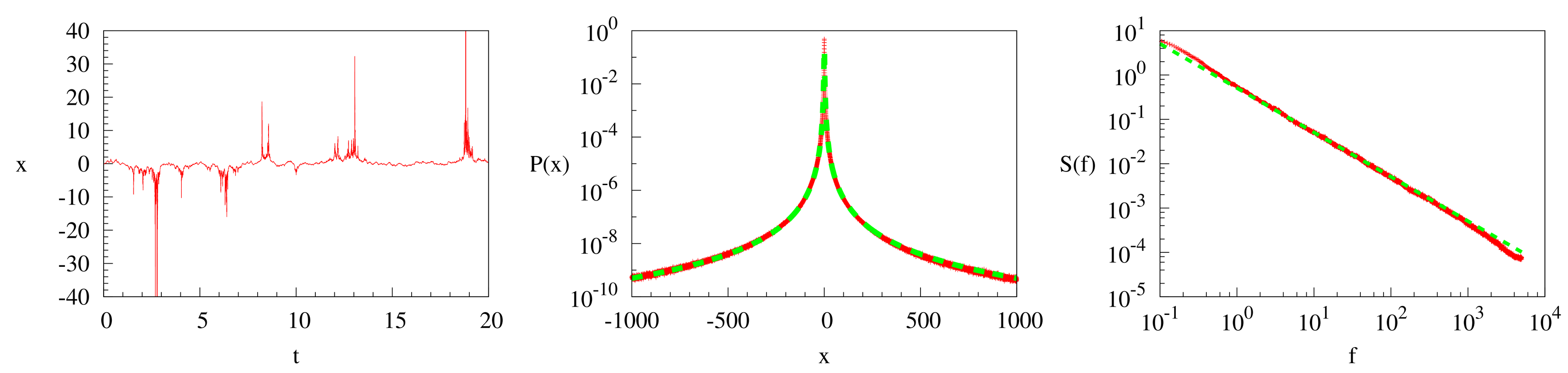

## $q$-Gaussian distribution

SDE

$$dx=\sigma^2\left(\eta-\frac{1}{2}\lambda\right)(x^2+x_0^2)^{\eta-1}xdt+\sigma(x^2+x_0^2)^{\eta/2}dW$$

Steady-state PDF:

$$P(x)=\frac{\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi}x_0\Gamma\left(\frac{\lambda-1}{2}\right)}\left(\frac{x_0^2}{x_0^2+x^2}\right)^{\frac{\lambda}{2}}=\frac{\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi}x_0\Gamma\left(\frac{\lambda-1}{2}\right)}\exp_q\left(-\lambda\frac{x^2}{2x_0^2}\right)\,,\qquad q=1+2/\lambda$$




## Superstatistics: $q$-exponential distribution

Local stationary PDF

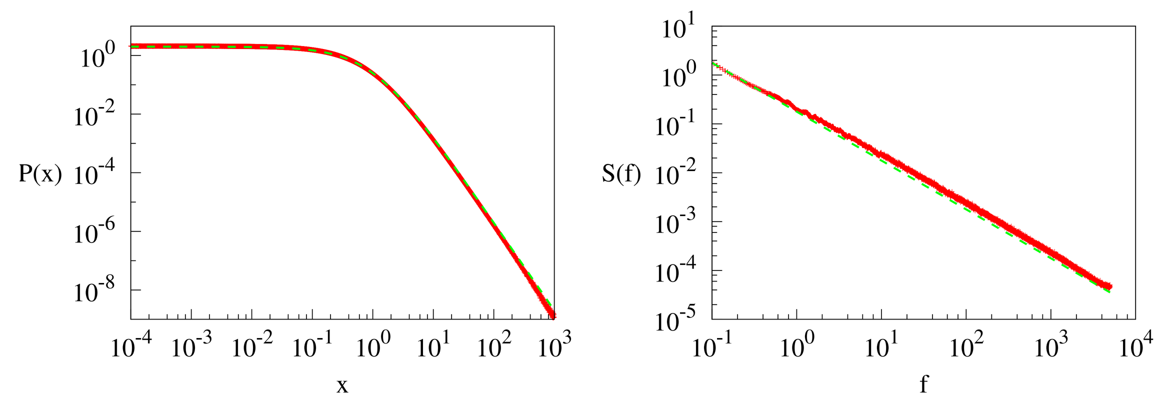
$$\varphi(x|\bar{x})=\bar{x}^{-1}\exp(-x/\bar{x})$$

The mean $\bar{x}$ obeys SDE

$$d\bar{x}=\sigma^2\left[\eta-\frac{\lambda}{2}+\frac{1}{2}\frac{x_0}{\bar{x}}-\frac{1}{2}\frac{\bar{x}}{\bar{x}_{\max}}\right]\bar{x}^{2\eta-1}dt+\sigma\bar{x}^\eta dW$$

The long-term stationary PDF of the signal $x$:

$$P(x)=\int_0^\infty\varphi(x|\bar{x})p(\bar{x})d\bar{x}=\frac{\lambda-1}{x_0}\left(\frac{x_0}{x+x_0}\right)^\lambda=\frac{\lambda-1}{x_0}\exp_q(-\lambda x/x_0)\,,\qquad q=1+1/\lambda$$




## Superstatistics: $q$-Gaussian distribution

Local stationary PDF

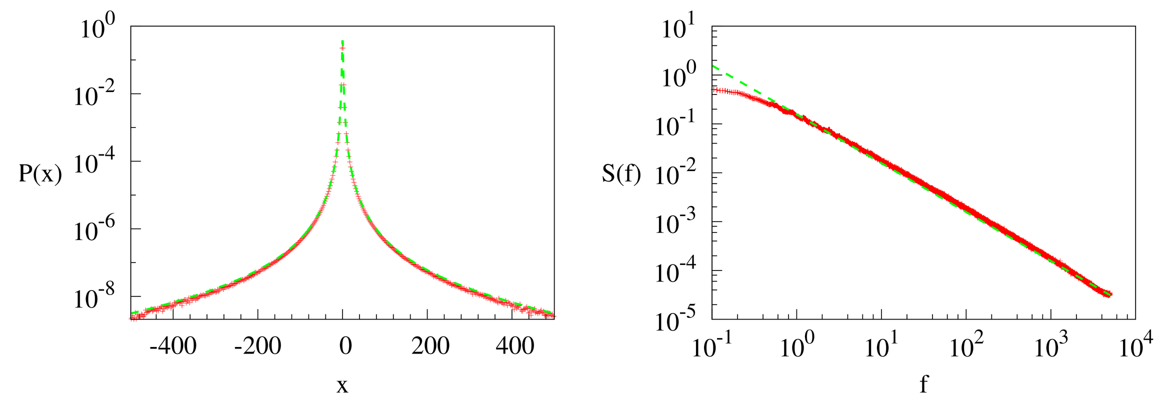
$$\varphi(x|\bar{x})=\frac{1}{\sqrt{\pi}\bar{x}}\exp(-x^2/\bar{x}^2)\,.$$

The fluctuating parameter $\bar{x}$ obeys SDE

$$d\bar{x}=\sigma^2\left[\eta-\frac{\lambda}{2}+\frac{x_0^2}{\bar{x}^2}-\frac{\bar{x}^2}{\bar{x}_{\max}^2}\right]\bar{x}^{2\eta-1}dt+\sigma\bar{x}^\eta dW$$

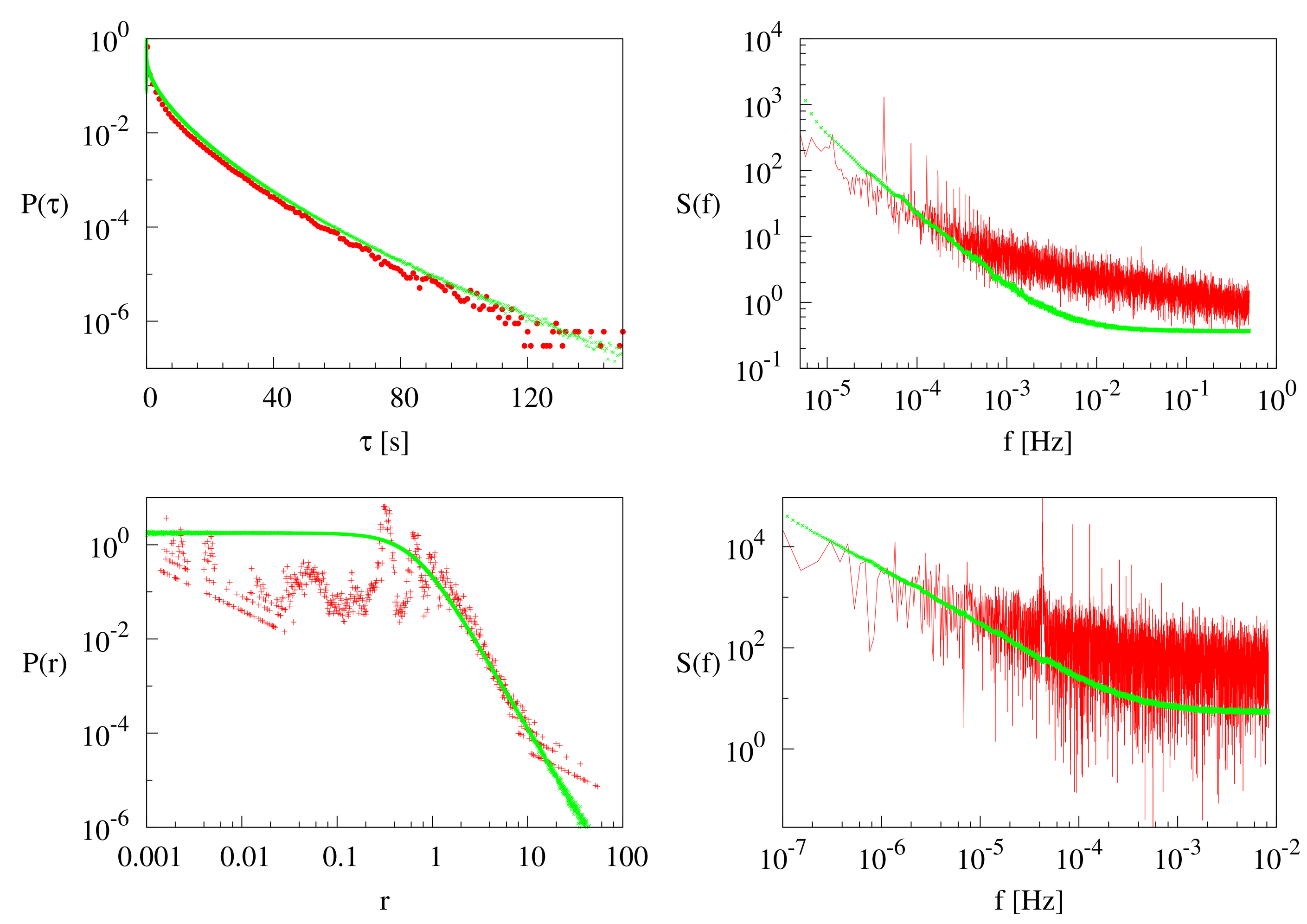
The long-term stationary PDF:

$$P(x)=\frac{\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi}x_0\Gamma\left(\frac{\lambda-1}{2}\right)}\left(\frac{x_0^2}{x_0^2+x^2}\right)^{\frac{\lambda}{2}}=\frac{\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi}x_0\Gamma\left(\frac{\lambda-1}{2}\right)}\exp_q\left(-\lambda\frac{x^2}{2x_0^2}\right)\,,\qquad q=1+2/\lambda$$




## Application for the modeling of trading

- Each individual intertrade duration $\tau$ is distributed according to exponential distribution with mean $1/n$.
- The trading activity $n$ changes in time according to non-linear SDE.
- Local PDF of return conditioned to value of the parameter $n$ is normal distribution with zero mean
- Standard deviation of the local PDF of return is proportional to trading activity $n$.




PDF and spectrum of intertrade duration $\tau$ and of the absolute value of the normalized one-minute return $r$

## Conclusions

- Both $1/f^\beta$ spectrum and $q$-Gaussian distribution can be obtained from the same non-linear SDE.
- The superstatistical framework using a fast dynamics with the slowly changing parameter described by nonlinear SDE can retain $1/f^\beta$ spectrum.
- The inter-trade durations, the trading activity and the normalized return may be replicated using the superstatistical approaches with the exponential and normal distributions of the local signals driven by the nonlinear stochastic process of the means

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