Intermittency generating $1/f$ noise

Julius Ruseckas, Bronislovas Kaulakys
Institute of Theoretical Physics and Astronomy, Vilnius University, Vilnius, Lithuania

Abstract

The phrase "$1/f$ noise" refers to the well-known empirical fact that in many systems at low frequencies the noise spectrum exhibits an approximately $1/f$ shape. Generating mechanisms leading to $1/f^{\beta}$ noise are still an open question. Here we analyze nonlinear dynamical systems with invariant subspace having the transverse Lyapunov exponent equal to zero. In particular, we explore nonlinear maps having power-law dependence on the deviation from the invariant subspace. We demonstrate that such maps can generate signals exhibiting $1/f^{\beta}$ noise and intermittent behavior. In contrast to known mechanism of $1/f$ noise involving Pomeau-Manneville type maps, coefficients in the maps we consider are not static, similarly as in the maps describing on-off intermittency. We relate the nonlinear dynamics described by proposed maps to $1/f$ noise models based on the nonlinear stochastic differential equations.

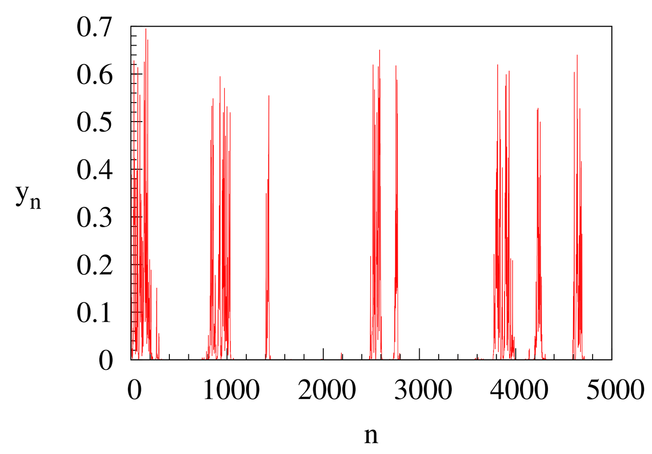
1. J. Ruseckas and B. Kaulakys, Chaos **23**, 023102 (2013).

Intermittency

Intermittency is an apparently random alternation of a signal between a quiescent state and bursts of activity.

Well known models of intermittency:

- Pomeau-Manneville intermittency
- Crisis-induced intermittency
- On-off intermittency



Model of intermittency with zero transverse Lyapunov exponent

Two-dimensional maps having a skew product structure:

$$x_{n+1} = F(x_n)\,, \qquad y_{n+1} = G(x_n, y_n)$$

$G(x,0)=0$, thus $y=0$ is the invariant subspace.
The dynamics $x_{n+1} = F(x_n)$ restricted to the invariant subspace $y=0$ is chaotic.
The two terms with the lowest powers in the expansion of the function $G(x,y)$ in the power series of y have the form

$$G(x,y) = y + g(x)y^{\eta}\,, \qquad \eta > 1$$

$\partial G(x,0)/\partial y = 1$ and, consequently, the transverse Lyapunov exponent is zero.
If $g(x)$ is not constant and can acquire both positive and negative values, the expansion leads to the the map for small values of y_n

$$y_{n+1} = y_n + z_n y_n^{\eta}\,, \qquad z_n \equiv g(x_n)\,, \qquad \eta > 1$$

If the average of the variable z_n is positive, $\langle z\rangle > 0$, and there is a global mechanism of reinjection, the map leads to the intermittent behavior.
In order to determine PDF of y and PSD of the series $\{y_n\}$, more terms in the expansion of the function $G(x,y)$ in the power series of y are needed

$$y_{n+1} = y_n + z_n y_n^{\eta} + \gamma y_n^{2\eta-1}$$

Another map:

$$y_{n+1} = (y_n^{1-\eta} + (1-\eta)z_n)^{\frac{1}{1-\eta}}$$

Approximation by stochastic differential equations

We replace the variable z_n by a random Gaussian variable having the same average and variance as z_n and interpret the map as Euler-Marujama approximation of a SDE

$$dy = \sigma^2\left(\eta - \frac{\nu}{2} + \frac{\eta-1}{2}\left(\frac{y_{\min}}{y}\right)^{\eta-1}\right) y^{2\eta-1}dt + \sigma y^{\eta} dW$$

Here

$$\sigma = \sqrt{\langle(z-\langle z\rangle)^2\rangle}\,, \qquad y_{\min} = \left[\frac{2\langle z\rangle}{(\eta-1)\langle(z-\langle z\rangle)^2\rangle}\right]^{\frac{1}{\eta-1}}\,, \qquad \nu = 2\eta - \frac{2\gamma}{\langle(z-\langle z\rangle)^2\rangle}$$

Approximation is valid when

$$y_{\max} \lesssim \langle(z-\langle z\rangle)^2\rangle^{-\frac{1}{2(\eta-1)}}$$

The condition $y_{\max}/y_{\min} \gg 1$ is obeyed when

$$\langle(z-\langle z\rangle)^2\rangle \gg \langle z\rangle^2$$

The SDE leads to the steady state PDF

$$P_0(y) = \frac{(\eta-1)y_{\min}^{\nu-1}}{\Gamma\left(\frac{\nu-1}{\eta-1}\right)y^{\nu}} \exp\left[-\left(\frac{y_{\min}}{y}\right)^{\eta-1}\right]$$

SDE generates signals with PSD having the form $S(f) \sim f^{-\beta}$ in a wide range of frequencies with the exponent

$$\beta = 1 + \frac{\nu-3}{2(\eta-1)}$$

Estimation of the range of frequencies where the PSD has the power-law form:

$$\left(\frac{y_{\min}}{y_{\max}}\right)^{2(\eta-1)} \ll 2\pi f \ll 1$$

Numerical examples

As a mechanism of reinjection we use a reflection at $y = 0.5$, leading to the map

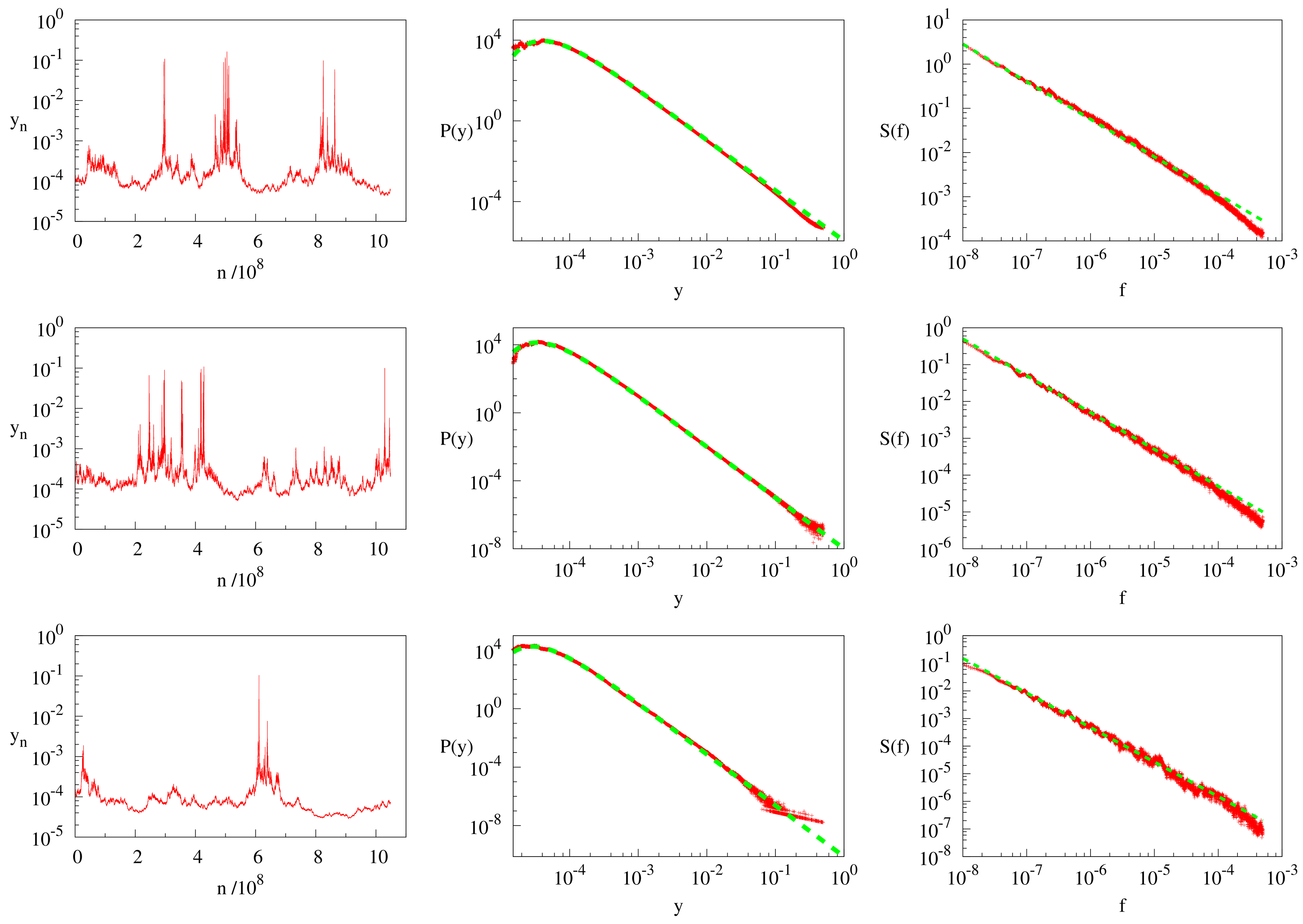
$$y_{n+1} = 0.5 - |y_n + z_n y_n^2 + \gamma y_n^3 - 0.5|$$

As a map $x_{n+1} = F(x_n)$ in we take the chaotic driving by a tent map

$$x_{n+1} = \begin{cases} 2x_n\,, & 0 \le x_n \le \frac{1}{2} \\ 2 - 2x_n\,, & \frac{1}{2} \le x_n \le 1. \end{cases}$$

The variable z_n with given average $\langle z\rangle$ and variance $\langle(z-\langle z\rangle)^2\rangle$ can be obtained from x_n using the equation

$$z_n = \sqrt{\frac{\langle(z-\langle z\rangle)^2\rangle}{\langle(x-\langle x\rangle)^2\rangle}}(x_n - \langle x\rangle) + \langle z\rangle$$



$\gamma = 0.75$ (upper row), $\gamma = 0.5$ (middle row), $\gamma = 0.25$ (lower row); $\langle z\rangle = 5\times10^{-5}$ and $\langle(z-\langle z\rangle)^2\rangle = 1$.
Another example

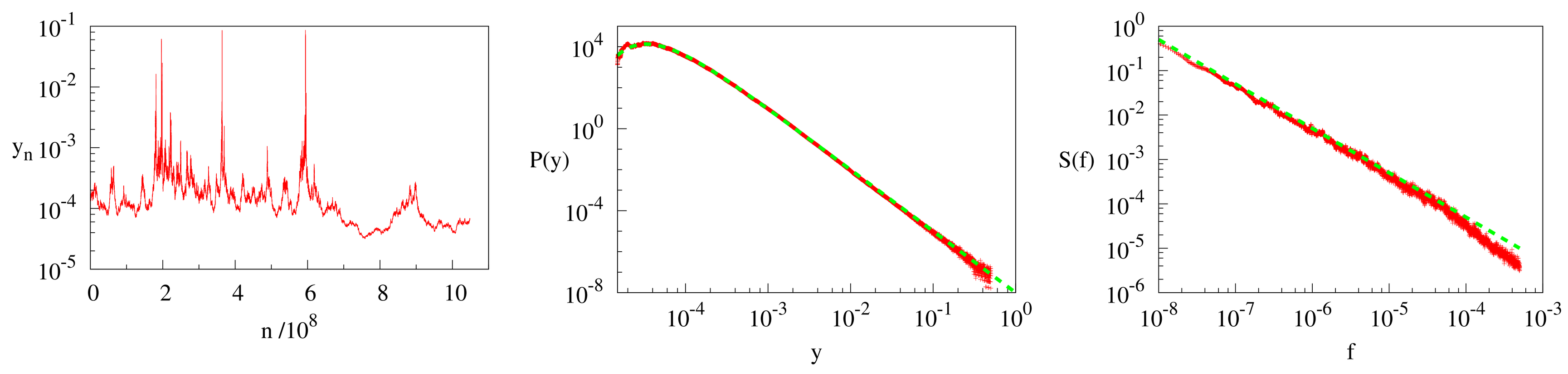
$$y_{n+1} = \begin{cases} y_n - y_n^2\zeta + 0.5y_n^3\,, & 0 \leqslant x_n \leqslant p_-\,, \\ 0.5 - |y_n + y_n^2\zeta + 0.5y_n^3 - 0.5|\,, & p_- < x_n \leqslant 1. \end{cases}$$

where

$$p_{\pm} = \frac{1}{2} \pm \frac{\langle z\rangle}{2\sqrt{\langle(z-\langle z\rangle)^2\rangle + \langle z\rangle^2}}$$

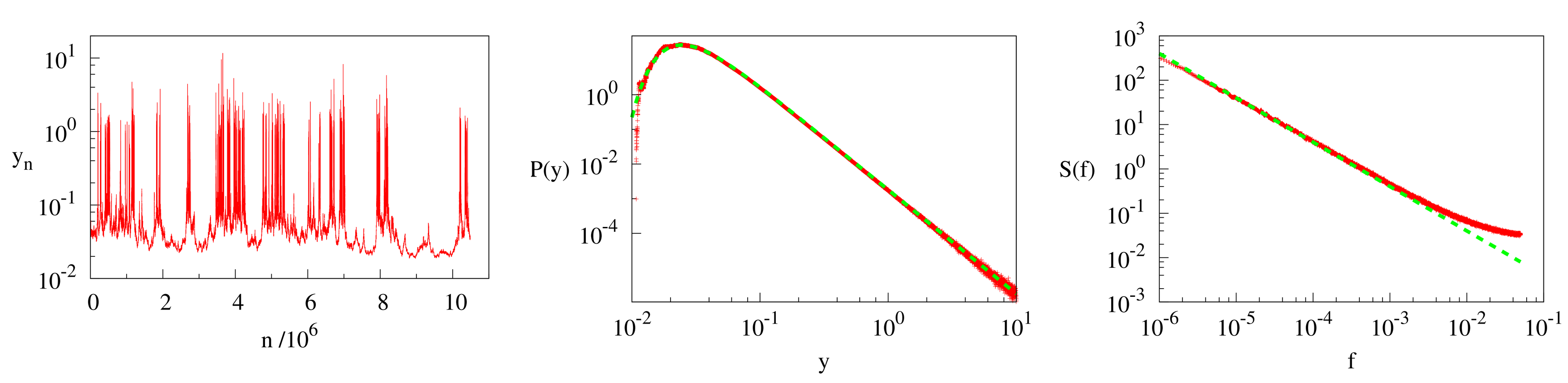
and

$$\zeta = \sqrt{\langle(z-\langle z\rangle)^2\rangle + \langle z\rangle^2}$$



Third example

$$y_{n+1} = \frac{1}{\sqrt{\left|\frac{1}{y_n^2} - 2z_n\right|}}$$



$\langle z\rangle = 9\times10^{-4}$

Conclusions

- The nonlinear maps having invariant subspace and the expansion in the powers of the deviation from the invariant subspace having the form $G(x,y) = y + g(x)y^{\eta}$ can generate signals with $1/f$ noise.
- In contrast to known mechanism of $1/f$ noise involving Pomeau-Manneville type maps, the parameter z_n in the map is not static.
- The exponent β in the PSD depends on two parameters η and ν, thus $1/f^{\beta}$ noise can be obtained for various values of the exponent β.
- The width of the frequency region where the PSD has $f^{-\beta}$ behavior is limited by the average value of the variable z_n: this width increases as $\langle z\rangle$ approaches the threshold value $\langle z\rangle = 0$.
- The width of the power-law region in the PSD increases with increasing the difference $\eta - 1$.

julius.ruseckas@tfai.vu.lt http://www.itpa.lt/∼ruseckas