Nonlinear stochastic differential equation as the background of financial fluctuations

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Abstract

We present nonlinear stochastic differential equation (SDE) which forms the background for the stochastic modeling of return in the nancial markets. SDE is obtained by the analogy with earlier proposed model of trading activity in the nancial markets and generalized within the nonextensive statistical mechanics framework. Proposed stochastic model generates time series of return with two, the probability distribution function and the power spectral density, power-law statistics.

Introduction

Recently we investigated the properties of stochastic multiplicative point processes analytically and numerically [1]. We derived formula for the power spectrum and related the model with the general form of multiplicative stochastic differential equations [2, 3]. Consequently, the stochastic model of trading activity based on the Poisson-like process driven by the nonlinear stochastic differential equation (SDE) was presented in Refs. [4, 5, 6, 7]. The statistical similarity of trading activity and absolute return together with the general background of non-extensive statistics give us an opportunity to model dynamics of return by nonlinear SDE.

Stochastic model with q-Gaussian PDF and long memory

We have proposed **nonlinear** SDE [2, 3]

$$\mathrm{d}x = \sigma^2\left(\eta - \frac{\lambda}{2}\right)x^{2\eta-1}\mathrm{d}t + \sigma x^{\eta}\mathrm{d}W \qquad (1)$$

Due to requirement of the stationarity of the process, the stochastic equation (1) should be analyzed together with the appropriate restrictions of the diffusion in some finite interval $x_{\min} \leqslant x \leqslant x_{\max}$ [2]. Solutions of SDE (1) exibit power-law probability distribution function, $P(x) \sim x^{-\lambda}$, and power spectral density, $S(f) \sim 1/f^{\beta}$, in wide range of frequencies. Here

$$\beta = 1 - \frac{\lambda - 3}{2\eta - 2}, \qquad \frac{1}{2} < \beta < 2\,. \qquad (2)$$

Power spectral density is determined mainly by the power-law behavior of the coefficients of SDE (1) at big values of $x \gg x_{\min}$. Changing the coefficients at small x, the spectrum retains power-law behavior. Therefore, we propose the following **modification** of Eq. (1)

$$\mathrm{d}x = \sigma^2\left(\eta - \frac{\lambda}{2}\right)(x_0^2 + x^2)^{\eta-1}x\mathrm{d}t + \sigma(x_0^2 + x^2)^{\eta/2}\mathrm{d}W. \qquad (3)$$

The associated Fokker-Planck equation gives q-Gaussian PDF,

$$P(x) = \frac{\Gamma(\lambda/2)}{\sqrt{\pi}x_0\Gamma((\lambda-1)/2)}\left(\frac{x_0^2}{x_0^2+x^2}\right)^{\lambda/2} \equiv A_q \exp_q\left(-\lambda\frac{x^2}{(3-q)\sigma_q^2}\right), \qquad (4)$$

$$q = 1 + 2/\lambda. \qquad (5)$$

Here $\exp_q(\cdot)$ is q-exponential defined as $\exp_q(x) \equiv (1 + (1-q)x)^{1/(1-q)}$.

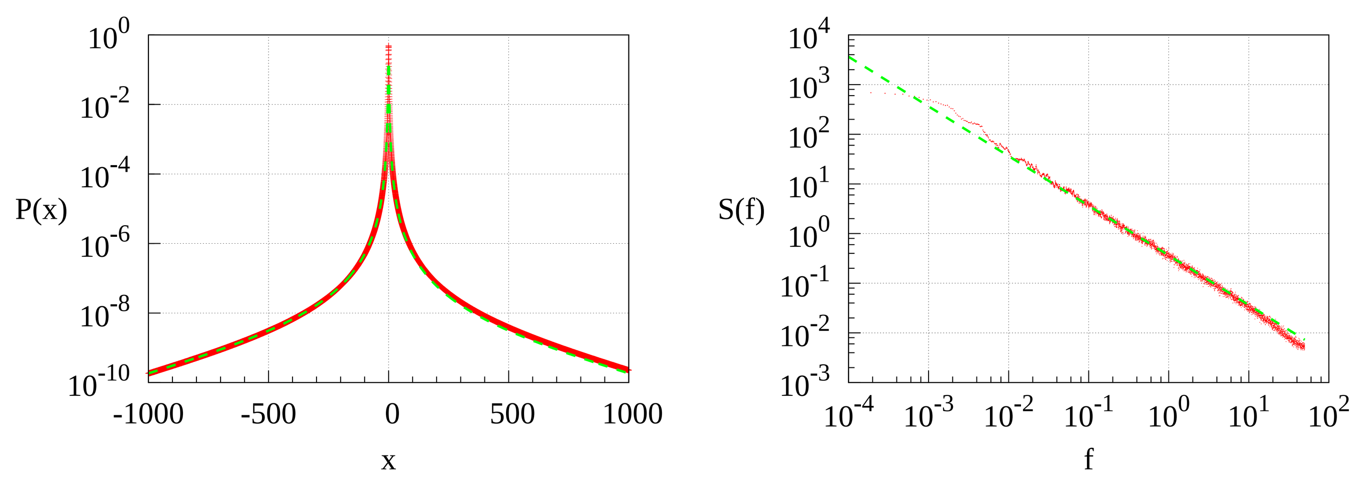

FIGURE 1: PDF, $P(x)$, of solution of Eq. (3) with parameters $x_0 = 1$, $\sigma = 1$, $\lambda = 3$ and $\eta = 5/2$ coinciding with the analytical Eq. (4) and spectrum, $S(f)$, in comparison with the line representing $1/f$ spectrum.

Equation with two power-law exponents

In order to get spectrum with two power-law exponents we propose SDE

$$\mathrm{d}\tilde{x} = \left(\eta - \frac{\lambda}{2} - (\tilde{x}\epsilon^{\eta})^2\right)\frac{(1+\tilde{x}^2)^{\eta-1}}{((1+\tilde{x}^2)^{1/2}\epsilon + 1)^2}\tilde{x}\mathrm{d}\tilde{t} + \frac{(1+\tilde{x}^2)^{\eta/2}}{(1+x^2)^{1/2}\epsilon + 1}\mathrm{d}\tilde{W}. \qquad (6)$$

Here $\tilde{x} = x/x_0$ and $\tilde{t} = t\sigma^2 x_0^{2(\eta-1)}$ are scaled variables. We demonstrate an example solutions of equation (6) in figure 2.

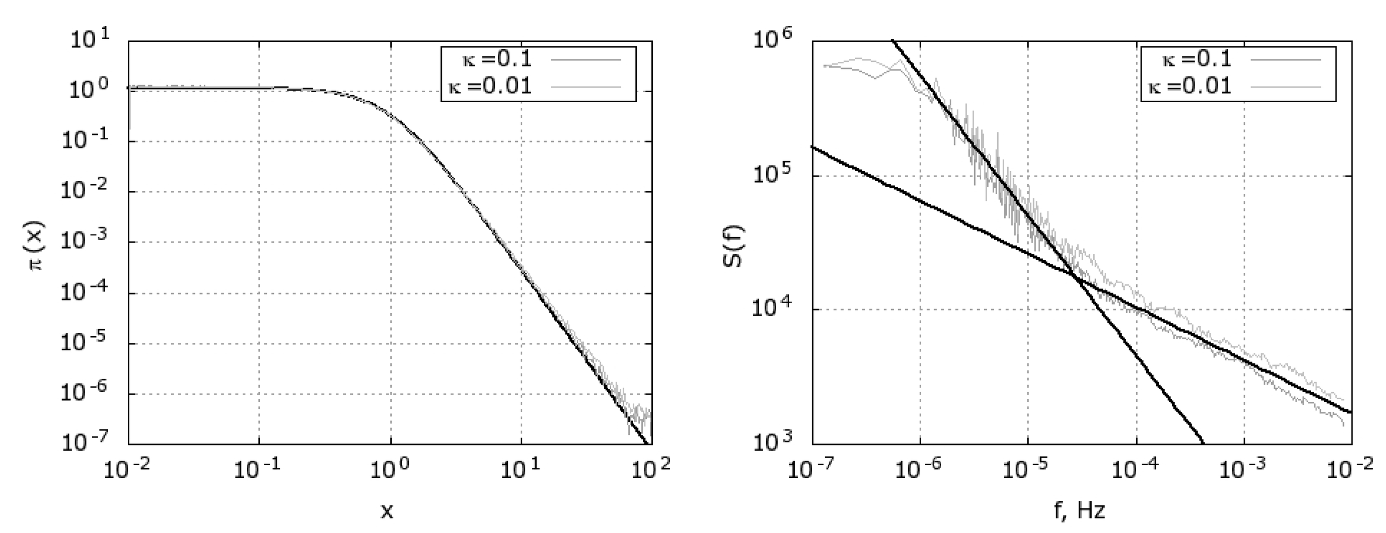

FIGURE 2: Model calculated from Eq. (6) with the parameters $\eta = 5/2$, $\lambda = 4.0$, and $\epsilon = 0.01$ PDF, $P(x)$, (continuum line) in comparison with empirical histogram of one minute returns of ABT stocks traded on NYSE (dots) and power spectrum, $S(x)$, of returns.

Power-law statistics of financial variables

We consider two statistics of two financial variables: probability distribution functions (PDF) and power spectral density $S(f)$.

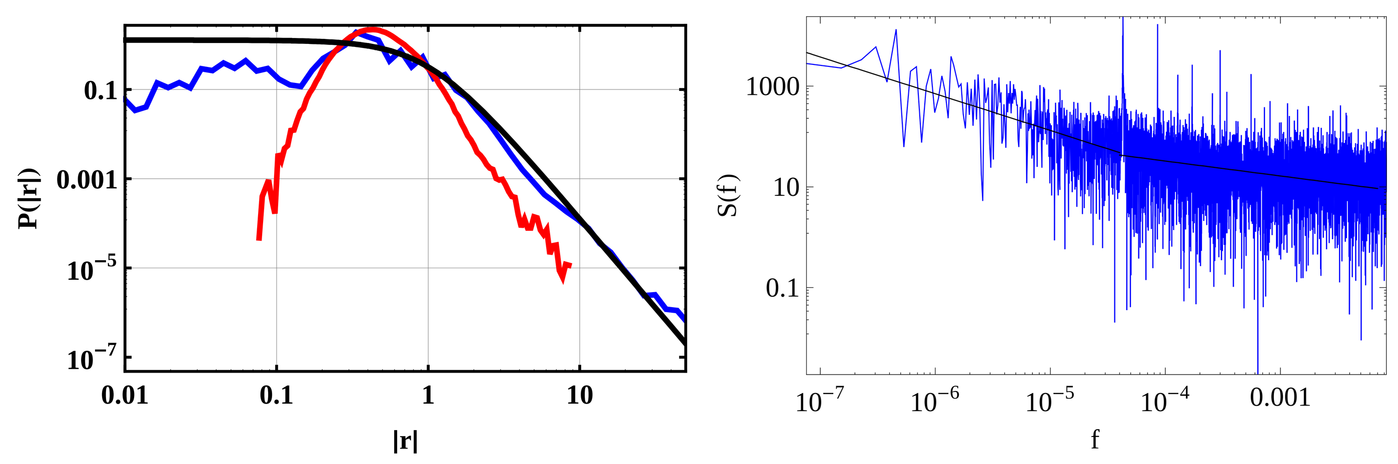

FIGURE 3: Left: PDF of absolute return, empirical (blue), moving averge (red), model (black). Right: power spectrum of the absolute return for IBM stocks, $\beta_1 = 0.3$, $\beta_2 = 0.73$.

Nonlinear stochastic model of return

We assume that empirical return r can be written as

$$r = \xi\{f(\mathrm{MA}(r)), \lambda_2\} \qquad (7)$$

Here $\xi\{r_0, \lambda\}$ is a q-Gaussian stochastic variable, and $\mathrm{MA}(r)$ a moving average of return. Conditional $\xi\{r_0, \lambda\}$ can be determined from empirical data for the fixed values of moving average $\mathrm{MA}(r)$. The q-Gaussians with $\lambda_2 = 5$ and

$$f(\mathrm{MA}(r)) = 1 + 2.5 \times |\mathrm{MA}(r)|$$

are good approximations of ξ fluctuations for all stocks and values of modulating $\mathrm{MA}(r)$. The PDF of moving average $\mathrm{MA}(r)$ can be approximated with q-Gaussian with parameters $\bar{r}_0 = 0.2$ and $\lambda = 4$. We propose to model the slowly diffusing long range memory modulating stochastic return $\mathrm{MA}(r)$ by $X = 1/\tau\int_t^{t+\tau} x(t')dt'$, where x is a continuous stochastic variable defined by nonlinear SDE (6):

$$\mathrm{d}\tilde{x} = \left(\eta - \frac{\lambda}{2} - (\tilde{x}\epsilon^{\eta})^2\right)\frac{(1+\tilde{x}^2)^{\eta-1}}{((1+\tilde{x}^2)^{1/2}\epsilon + 1)^2}\tilde{x}\mathrm{d}\tilde{t} + \frac{(1+\tilde{x}^2)^{\eta/2}}{(1+x^2)^{1/2}\epsilon + 1}\mathrm{d}\tilde{W}.$$

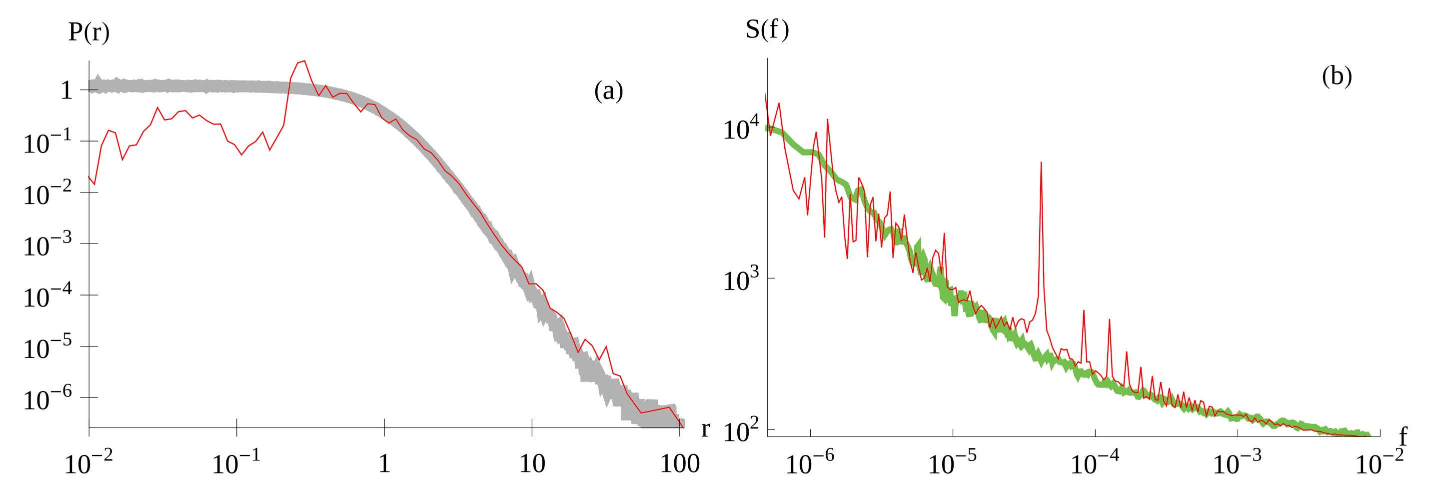

FIGURE 4: Comparison of empirical and model statistics of one minute returns traded on NYSE, PDF of normalized returns (a) empirical (red thin line) and model (gray thick line); and power spectrum of absolute return (b) empirical (red thin line) averaged over 24 stocks and model (green thick line) averaged over 24 realizations. All parameters are as follows: $\lambda = 3.6$ $q_2 = 1.4$; $\bar{r}_0 = 0.2$; $\tau = 0.0001/\sigma^2 = 60\,\mathrm{s}$; $\epsilon = 0.01$.

Conclusions

We propose the nonlinear SDE reproducing the fascinating statistical properties of the financial variables with q-Gaussian PDF and fractured behavior of the power spectrum. The proposed stochastic model with empirically defined parameters reproduces the distribution of return and the correlations evaluated through the power spectral density of absolute return. Stochastic modeling of the financial variables by nonlinear SDE is consistent with the nonextensive statistical mechanics and provides new opportunities to capture empirical statistics in detail.

Acknowledgements

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