Power-law statistics from nonlinear stochastic differential equations driven by Lévy stable noise

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Outline

Introduction: SDEs with Lévy noise

SDEs with Lévy noise generating signals having power-law PDF

- signals acquiring only positive values
- signals acquiring both positive and negative values

Power spectral density of the generated signals

Summary

Lévy α-stable distribution

- Stable distributions are "attractors" for sums of independent and identically distributed random variables.
- Lévy stable distribution is a result of generalized central limit theorem.
- Characteristic function has the form

$$\mathcal{F}\left[P_\alpha(z)\right] = \langle \exp(\mathrm{i}kz) \rangle = \exp(-\sigma^\alpha |k|^\alpha)\,.$$

- Asymptotically power-law:

$$P_\alpha(z) \sim 1/z^{1+\alpha}$$

Lévy α-stable distribution

Special cases:

- $\alpha = 1$ Cauchy distribution

$$P_1(z) = \frac{\sigma}{\pi(\sigma^2 + z^2)}$$

- $\alpha = 3/2$ Holtsmark distribution
- $\alpha = 2$ Gaussian distribution

$$P_2(z) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{z^2}{2\sigma}}$$

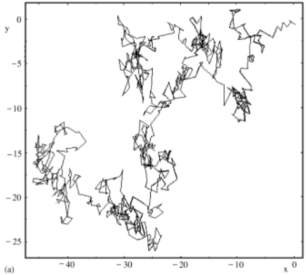
Lévy α-stable distribution

There are systems exhibiting Lévy α-stable distributions:

- turbulent magnetised plasma
- photons in hot atomic vapours
- velocity distribution of particles in fractal turbulence

Brownian motion and Lévy flight

V.V. Yanovsky et al. / Physica A 282 (2000) 13–34



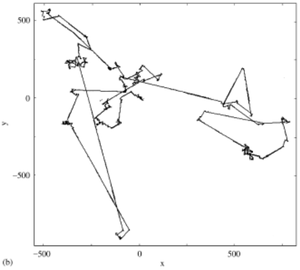


Fig. 1. An illustration of the Brownian motion which corresponds to normal diffusion. It is merely obtained by iterating any random walk with identically independently distributed elementary steps having finite variance. Here are displayed the first 1000 (a), 3000 (b) steps of the walk.

Fig. 2. An illustration of a Lévy motion, often called Lévy flight. Contrary to the Brownian motion, not only the variance of the elementary steps are infinite, but also any moment of order $q \geqslant \alpha$, where the Lévy index α is the most important of the four parameters characterizing this process ($\alpha = 1.5$ in the present case). As

Lévy flights modeled by SDEs with Lévy noise

We consider the Langevin equation of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(x) + b(x)\xi(t)$$

The stochastic force $\xi(t)$ is:

- uncorrelated, $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$
- characterized by Lévy α-stable distribution: $\langle \exp(\mathrm{i}k\xi) \rangle = \exp(-\sigma^\alpha |k|^\alpha)$
- α is the index of stability and σ is the scale parameter.

Another way to write the same equation:

$$\mathrm{d}x = a(x)\mathrm{d}t + b(x)\mathrm{d}L_t^\alpha$$

$\mathrm{d}L_t^\alpha$ stands for the increments of Lévy α-stable motion L_t^α

Fractional Fokker-Planck equation

Fractional Fokker-Planck equation instead of stochastic differential equation:

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}a(x)P(x,t) + \sigma^\alpha \frac{\partial^\alpha}{\partial |x|^\alpha} b(x)^\alpha P(x,t)$$

Here $\partial^\alpha / \partial |x|^\alpha$ is the Riesz-Weyl fractional derivative:

$$\mathcal{F}\left[\frac{\partial^\alpha}{\partial |x|^\alpha} f(x)\right] = -|k|^\alpha \tilde{f}(k)$$

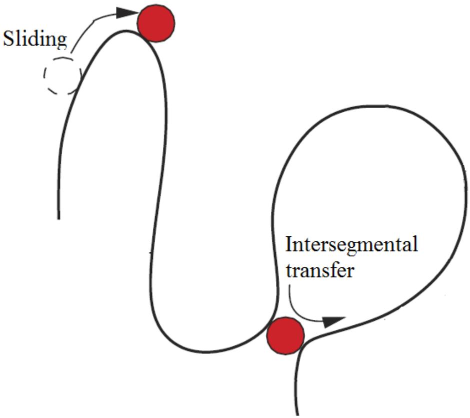
Fractional equations in physical systems

If $a(x) = 0$ and $b(x) = \mathrm{const}$

$$\frac{\partial}{\partial t} P(x,t) = \sigma^\alpha \frac{\partial^\alpha}{\partial |x|^\alpha} P(x,t)$$

Diffusion on fractal structures:

- polymers
- plasmas
- fractal turbulence in liquids



Goals

1. To find a simple stochastic differential equation with Lévy stable noise generating signals having power-law steady state PDF,

$$P_0(x) \sim x^{-\lambda}$$

2. To generalize previously proposed nonlinear stochastic differential equation with Gaussian noise that generates signals having $1/f$ spectrum

$1/f$ noise

- In contrast to the Brownian motion, the signals and processes with $1/f$ spectrum cannot be modeled by the linear stochastic equations
- A nonlinear stochastic differential equation

$$\mathrm{d}x = \sigma^2(\eta - \lambda/2)x^{2\eta-1}\mathrm{d}t + \sigma x^{\eta}\mathrm{d}W_t$$

  generates signals exhibiting $1/f$ spectrum

  B. Kaulakys and J. Ruseckas, Phys. Rev. E **70**, 020101(R) (2004).

  B. Kaulakys and J. Ruseckas, V. Gontis, and M. Alaburda, Physica A **365**, 217 (2006).

- Has been used to describe signals in socio-economical systems

Problem

- The Riesz-Weyl fractional derivative is complicated. In the coordinate space

$$\frac{\partial^\alpha}{\partial|x|^\alpha} f(x) = -\frac{1}{2\cos\left(\frac{\pi\alpha}{2}\right)} \{D_+^{-\alpha} f(x) + D_-^{-\alpha} f(x)\}$$

  where $D_+^{-\alpha}$ and $D_-^{-\alpha}$ are the left and right Riemann-Liouville derivatives
- It is difficult to obtain SDEs with Lévy noise having desired properties

Equation for steady state PDF

Equation for steady state PDF:

$$\sigma^{\alpha}\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}b(x)^{\alpha}P_0(x)-\frac{\partial}{\partial x}a(x)P_0(x)=0$$

- Can be written as $-\mathrm{d}J(x)/\mathrm{d}x = 0$, where $J(x)$ is the probability current.
- Reflective boundaries lead to the boundary condition $J(x) = 0$

SDEs generating signals with power-law PDF

Let us assume that

$$b(x) = x^{\eta}$$

Fractional derivative of the power-law function

$$f(x) = x^{\rho}, \qquad x_{\mathrm{min}} \ll x \ll x_{\mathrm{max}}$$

is

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}|x|^{\alpha}} f(x) \approx \frac{\sin\left(\pi\left(\frac{\alpha}{2} - \rho\right)\right)}{\sin\left(\frac{\pi}{2}(\rho - \alpha)\right)} \frac{\Gamma(1+\rho)}{\Gamma(1+\rho-\alpha)} x^{\rho-\alpha}, \qquad -1 < \rho < \alpha$$

SDEs generating signals with power-law PDF

When

$$b(x) = x^{\eta}$$

then the drift term has the form

$$a(x) = \sigma^{\alpha} \gamma x^{\mu}$$

where

$$\mu = \alpha(\eta - 1) + 1$$

and

$$\gamma = \frac{\sin\left[\pi\left(\frac{\alpha}{2} - \alpha\eta + \lambda\right)\right]}{\sin[\pi(\alpha(\eta - 1) - \lambda)]} \frac{\Gamma(\alpha\eta - \lambda + 1)}{\Gamma(\alpha(\eta - 1) - \lambda + 2)}$$

SDEs generating signals with power-law PDF

Proposed equation

$$\mathrm{d}x = \sigma^{\alpha}\gamma x^{\alpha(\eta-1)+1}\mathrm{d}t + x^{\eta}\mathrm{d}L_t^{\alpha}$$

Particular cases:

- $\alpha = 2$:
$$\mathrm{d}x = \sigma^2(2\eta - \lambda)x^{2\eta-1}\mathrm{d}t + x^{\eta}\mathrm{d}L_t^2 \,.$$
- $\alpha = 1$:
$$\mathrm{d}x = \sigma \cot[\pi(\lambda - \eta)]x^{\eta}\mathrm{d}t + x^{\eta}\mathrm{d}L_t^1 \,.$$

What if we allow x to be negative?

Power-law steady state PDF:

$$P_0(x) \sim |x|^{-\lambda}$$

Again we assume

$$b(x) = |x|^{\eta}\,, \qquad x \gg x_{\min}$$

Fractional derivative of the power-law function

$$f(x) = |x|^{\rho}, \qquad x_{\min} \ll |x| \ll x_{\max}$$

is

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}|x|^{\alpha}} f(x) \approx \frac{\sin\left(\frac{\pi}{2}\rho\right)}{\sin\left(\frac{\pi}{2}(\alpha-\rho)\right)} \frac{\Gamma(1+\rho)}{\Gamma(1+\rho-\alpha)} |x|^{\rho-\alpha}\,, \qquad -1 < \rho < \alpha$$

SDEs generating signals with power-law PDF

The drift term has the form

$$a(x) = \sigma^{\alpha} \gamma |x|^{\mu - 1} x$$

where

$$\mu = \alpha(\eta - 1) + 1$$

and

$$\gamma = \frac{\sin\left[\frac{\pi}{2}(\alpha\eta - \lambda)\right]}{\sin\left[\frac{\pi}{2}(\lambda - \alpha(\eta - 1))\right]} \frac{\Gamma(\alpha\eta - \lambda + 1)}{\Gamma(\alpha(\eta - 1) - \lambda + 2)}$$

γ has a different expression!

SDEs generating signals with power-law PDF

Proposed equations

$$\mathrm{d}x = \sigma^{\alpha}\gamma(x_0^2 + x^2)^{\frac{\alpha}{2}(\eta-1)}x\mathrm{d}t + (x_0^2 + x^2)^{\frac{\eta}{2}}\mathrm{d}L_t^{\alpha}$$

$$\mathrm{d}x = \sigma^{\alpha}\gamma(x_0^{\alpha} + |x|^{\alpha})^{\eta-1}x\mathrm{d}t + (x_0^{\alpha} + |x|^{\alpha})^{\frac{\eta}{\alpha}}\mathrm{d}L_t^{\alpha}$$

Estimation of power spectral density

- Autocorrelation function can be written as

$$C(t) = \int \mathrm{d}x \int \mathrm{d}x' \, xx' P_0(x) P_x(x', t|x, 0)$$

- $P_0(x)$ is the steady state PDF
- $P_x(x', t|x, 0)$ is the transition probability
- The transition probability can be obtained from the solution of the fractional Fokker-Planck equation with the initial condition $P_x(x', 0|x, 0) = \delta(x' - x)$.

Scaling property of the proposed equation

- Our equation:

$$\mathrm{d}x = \sigma^{\alpha}\gamma x^{\alpha(\eta-1)+1}\mathrm{d}t + x^{\eta}\mathrm{d}L_t^{\alpha}$$

- The increments of Lévy α-stable motion dL_t^{α} have the scaling property

$$\mathrm{d}L_{at}^{\alpha} = a^{1/\alpha}\mathrm{d}L_t^{\alpha}$$

- Changing the variable x to the scaled variable $x_s = ax$ or introducing the scaled time $t_s = a^{\alpha(\eta-1)}t$ one gets the same resulting equation.

Estimation of power spectral density

- Trasnsition probability has a scaling property

$$P_x(ax', t|ax, 0) = a^{-1} P_x(x', a^{\alpha(\eta-1)} t|x, 0)$$

- Steady state PDF has power-law form

$$P_0(x) \sim x^{-\lambda}$$

- Autocorrelation function $C(t)$ has scaling property

$$C(at) \sim a^{\beta-1} C(t)$$

  where

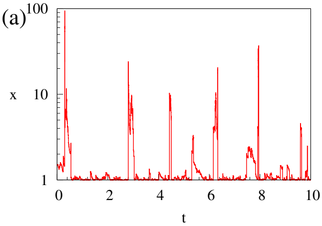
$$\beta = 1 + \frac{\lambda - 3}{\alpha(\eta - 1)}$$

- This means that $S(f) \sim f^{-\beta}$

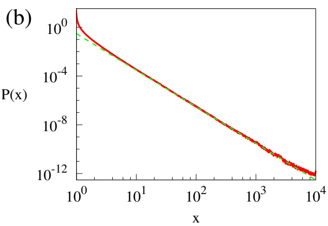
$1/f$ noise

- $S(f) \sim f^{-\beta}$ with
$$\beta = 1 + \frac{\lambda - 3}{\alpha(\eta - 1)}$$
- If $\lambda = 3$ then $\beta = 1$: we get $1/f$ noise
- Proposed stochastic differential equations with Lévy stable noise are yet another model generating signals having $1/f$ spectrum in a wide range of frequencies.

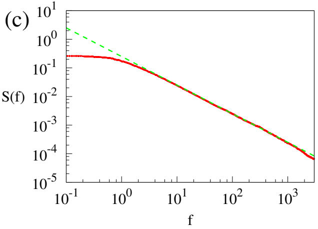
Numerical example: x only positive



Typical signal



Distribution of x



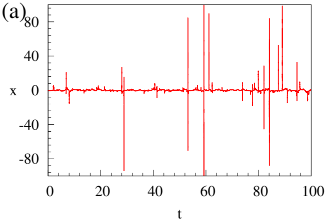
Power spectral density

$$\mathrm{d}x = -\sigma x^{9/4}\mathrm{d}t + x^{9/4}\mathrm{d}L_t^1$$

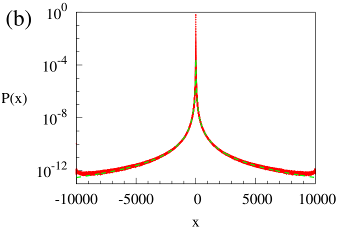
$\lambda = 3$, $\eta = 9/4$, $x_{\min} = 1$, $x_{\max} = 10^4$, $\sigma = 1$

$1/f$ spectrum

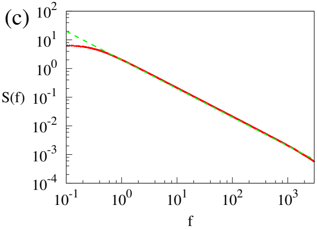
Numerical example: x positive and negative



Typical signal



Distribution of x



Power spectral density

$$\mathrm{d}x = -\sigma(x_0^2+x^2)^{3/4}x\,\mathrm{d}t+(x_0^2+x^2)^{5/4}\mathrm{d}L_t^1$$

$\lambda = 3$, $\eta = 5/2$, $x_0 = 1$, $x_{\max} = 10^4$, $\sigma = 1$

$1/f$ spectrum

Summary

- We obtain a class of nonlinear SDEs with Lévy stable noise giving the power-law steady state distribution of the signal intensity
- and power-law behavior of the power spectral density in any desirably wide range of frequencies.
- Proposed SDEs describe Lévy flights in non-equilibrium and non-homogeneous environments
- In contrast to the SDEs with the Gaussian noise, the constant in the drift term is different when the stochastic variable can also be negative.

Summary

- Replacing a Gaussian noise with a Lévy stable noise in the equation changes the scaling properties of the signal
- In order to preserve original scaling properties, the drift or diffusion coefficients should be changed as well.

Thank you for your attention!

R. Kazakevičius, J. Ruseckas, Physica A **411**, 95 (2014)