Continuous transition from the extensive to the non-extensive statistics in a simple model

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Outline

Motivation

Proposed model
- Mesoscopic description
- New network formation model

Properties of the model

Summary

Motivation

- Systems with long-range interactions:
  - gravitational forces
  - Coulomb forces in globally charged systems
  - vortices in two-dimensional fluid mechanics
  - trapped charged particles
- Violate extensivity and additivity,
  basic properties used to derive the thermodynamics of a system

Systems with long-range interactions

Exhibit unusual behavior:

- inequivalence of the microcanonical and canonical ensembles
- negative microcanonical specific heat
- slow relaxation
- broken ergodicity
- long-lived non-equilibrium quasistationary states (QSS)
- violent relaxation into these states
- non-Gaussian distributions
- non-exponential relaxations for autocorrelations

Non-extensive statistical mechanics

- Intended to describe some of the systems with long-range interactions by generalizing the Boltzmann-Gibbs statistics
- Based on a generalized entropy

$$S_q = \frac{1 - \int [p(x)]^q \, \mathrm{d}x}{q - 1}$$

Non-extensive statistical mechanics

Maximization of q-entropy with appropriate constraints leads to a q-Gaussian distribution

$$p(x) = C_q \exp_q(-A_q x^2)$$

Here

$$\exp_q(x) \equiv [1 + (1 - q)x]_+^{\frac{1}{1-q}}$$

is the q-exponential function

Systems with long-range interactions

- Mathematical description is usually complicated.
- Simple models providing some degree of intuition about non-extensive statistical mechanics can be useful for understanding it

Goal

To construct a simple model with long-range interactions.

Proposed model

- N particles of spin-$\frac{1}{2}$
- Each particle interacts with some number of other particles
- Dynamics is described as a Markov process in a continuous time
- The transition rate for a particle i having the spin X to flip it's spin is

$$p_i(X \to Y) = \sigma + hn_i(Y)$$

Here $n_i(Y)$ is the number of particles it interacts with, having the opposite spin Y

Mean-field approximation

Mean transition rate

$$\langle p_i(X \to Y)\rangle = \sigma + h\langle d\rangle \frac{N_Y}{N}$$

Here N_Y is the total number of particles with spin Y and $\langle d\rangle$ is the average number of particles interacting with a given particle.
Transition rates of one-step stochastic process:

$$p(n \to n+1) = (N-n)\,\eta_1(n,N)$$
$$p(n \to n-1) = n\,\eta_2(n,N)$$

where n is the number of particles with the spin pointing up and

$$\eta_1 = \langle p_i(\downarrow\to\uparrow)\rangle \qquad \eta_2 = \langle p_i(\uparrow\to\downarrow)\rangle$$

represent per-particle transition rates

Mean-field approximation

Transition rates:

$$\eta_1 = \sigma + \frac{h'}{N} n$$
$$\eta_2 = \sigma + \frac{h'}{N}(N - n)$$

The term with $h' = h\langle d \rangle$ represents interactions between particles

Mesoscopic description of the model

Fokker-Planck equation in the limit of large N:

$$\frac{\partial}{\partial t}P_x(x,t) = \frac{\partial}{\partial x}[x\eta_2 - (1-x)\eta_1]P_x(x,t) + \frac{1}{2N}\frac{\partial^2}{\partial x^2}[(1-x)\eta_1 + x\eta_2]P_x(x,t)$$

Here $x = \frac{n}{N}$

Short-range interactions

- Transition rates η_1 and η_2 depend only on $x = n/N$
- In the thermodynamic limit $N \to \infty$, the diffusion term becomes negligible
- Detailed balance:

$$x_0\, \eta_2(x_0) = (1 - x_0)\, \eta_1(x_0)\,.$$

- Gaussian distribution

$$\sqrt{\frac{NA}{\pi}} \exp\left[-NA(x - x_0)^2\right]$$

- The width is proportional to $1/\sqrt{N}$

Conclusion

To represent long-range interactions, the transition rates η_1 and η_2 should explicitly depend on N.

Proposed model

We assume that

$$\langle d \rangle \sim N^{\alpha}$$

with $0 \leq \alpha \leq 1$

Network of interactions

Interaction between particles can be represented by a network.
$\langle d \rangle$ is the mean degree, it should scale with the number of nodes as N^α

Known networks, important in social systems:

- Random network, $\langle d \rangle \sim N$
- Scale-free network, $\langle d \rangle \sim \mathrm{const}$

Conclusion

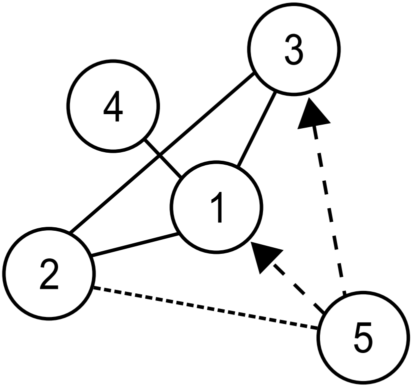
We need a network intermediate between a random network and a scale-free network

New network formation model

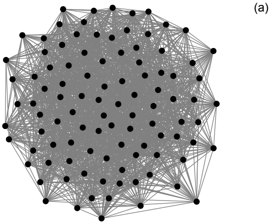
1. Add a new node to the network
2. Connect the new node to another node with the probability proportional to its degree.
3. Add additional connections to the immediate neighbours of the node in the step 2 with the probability

$$p = p_0 d^{-\gamma},$$

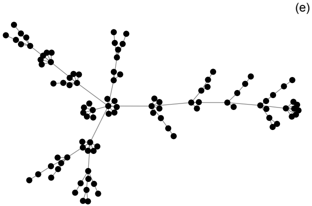
where d is a degree of the node in the step 2.



New network formation model

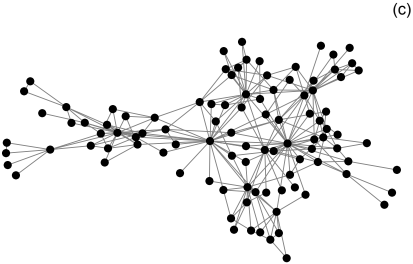


$\gamma = 0$

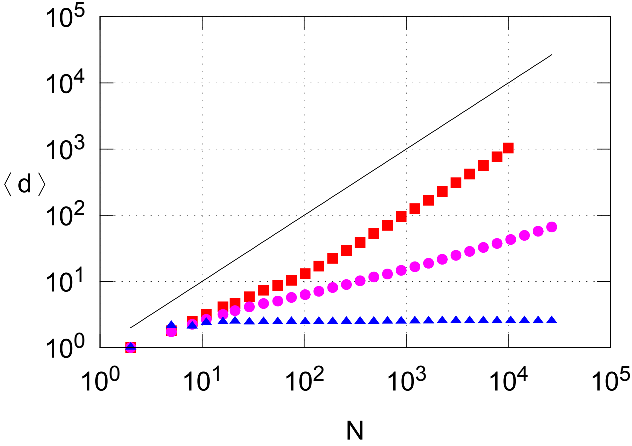


$\gamma = 1$

$\gamma = 0.3$



Scaling of the mean degree



$$\langle d \rangle \sim N^{\alpha} \qquad \alpha \approx (1-\gamma)^2$$

Steady state

Steady state distribution is a q-Gaussian

$$P_0(x) = C' \exp_q \left[-A_q \left(x - \frac{1}{2} \right)^2 \right]$$

with

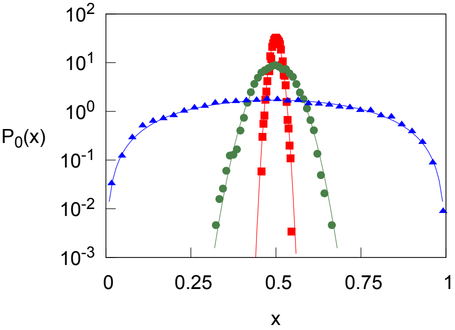
$$q = 1 - \frac{1}{\varepsilon N^{1-\alpha} - 1}, \qquad A_q = 2N^{1-\alpha} \frac{1 - \frac{1}{\varepsilon} N^{\alpha-1}}{\frac{1}{2\varepsilon} + N^{-\alpha}}$$

Here

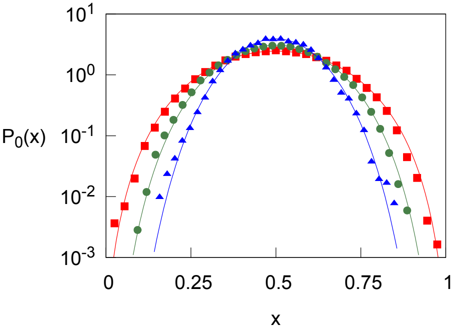
$$\varepsilon = \frac{\sigma}{d_0 h}$$

with $d = d_0 N^\alpha$

Steady state



Steady state distribution for different values of α



Steady state distribution for different values of N

Steady state: long-range interactions

If the interactions are long-range, $\alpha = 1$, the steady state distribution in the limit $N \to \infty$ does not depend on N

$$P_0(x) \sim [x(1-x)]^{\varepsilon - 1}$$

Non-extensivity parameter

$$q = 1 - \frac{1}{\varepsilon - 1}$$

Steady state: $\alpha < 1$

When $\alpha < 1$, in the limit of large N the steady state distribution is approximately Gaussian

$$P_0(x) \sim \exp\left[-N^{1-\alpha} A \left(x - \frac{1}{2}\right)^2\right]$$

with

$$A = \begin{cases} \frac{2}{\frac{1}{2\varepsilon}+1}\,, & \alpha = 0 \\ 4\varepsilon\,, & 0 < \alpha < 1 \end{cases}$$

When $\alpha > 0$, the fluctuations decay not as $1/\sqrt{N}$ but as $1/\sqrt{N^{\alpha-1}}$

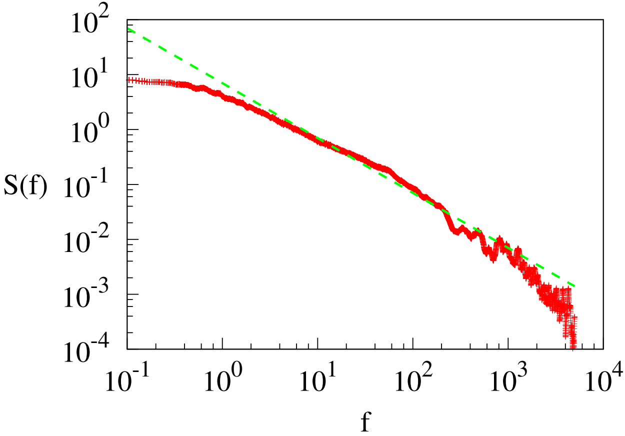
Fluctuation in time

Langevin equation corresponding to the Fokker-Planck equation:

$$\mathrm{d}x = \sigma(1-2x)\mathrm{d}t + \sqrt{\frac{1}{N}(2h\langle d\rangle x(1-x)+\sigma)}\mathrm{d}W_t$$

- When $\alpha < 1$, in the limit of $N \to \infty$, macroscopic fluctuations of x vanish
- When $\alpha = 1$, the fluctuations of the ratio $N_\downarrow/N_\uparrow$ have $1/f^\beta$ power spectral density in a wide region of frequencies, growing with N

$1/f$ noise



We have $1/f$ noise when $\frac{\sigma}{d_0 h} = 2$

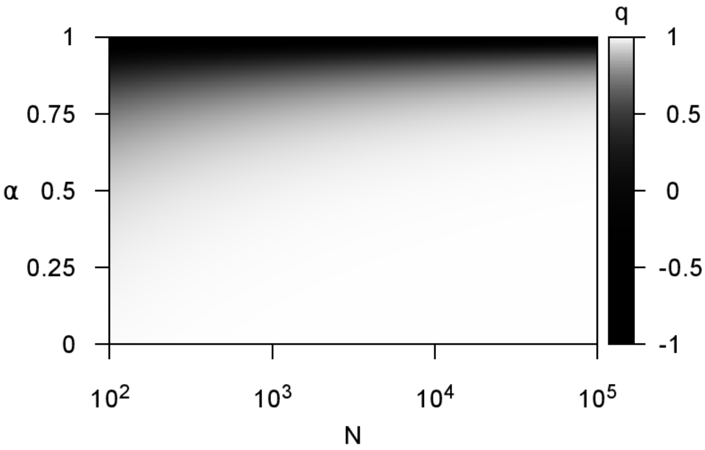
$q > 1$?

It can be shown that $q > 1$ only when $h < 0$
However, this leads either to

1. $q \to 1$ as $N \to \infty$
2. Model is valid only when

$$N < \frac{2-q}{q-1}$$

Transition from extensive to non-extensive statistics



Summary

- The steady state distribution for a finite system size is described by q-Gaussian
- This confirms the similarity of small systems to large systems with truly long-range interactions
- The extensive behaviour increases with increase of N
- For $\alpha = 1$ the model is non-extensive for all N
- The fluctuations decay more slowly with increasing N for larger values of α
- Proposed model cannot give $q > 1$ for large N

Thank you for your attention!