Modeling Tsallis distributions by nonlinear stochastic differential equations with application to financial markets

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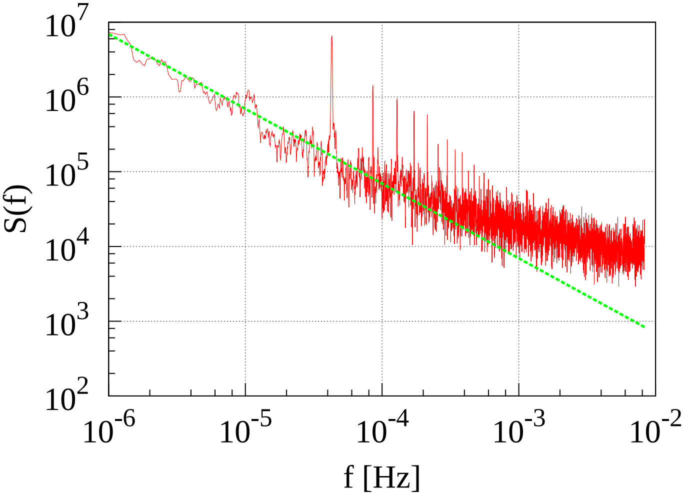
Outline

$1/f$ noise

Our researches are related with the $1/f$ noise problem and long-range processes.

- $1/f$ noise is a type of noise whose power spectral density $S(f)$ as a function of the frequency f behaves like $S(f) \sim 1/f^{\beta}$ where the exponent β is close to 1.
- Fluctuations of signals exhibiting $1/f$ behavior of the power spectral density at low frequencies have been observed in a wide variety of physical, geophysical, biological, financial, traffic, the Internet, astrophysical and other systems.

Example of $1/f$ noise

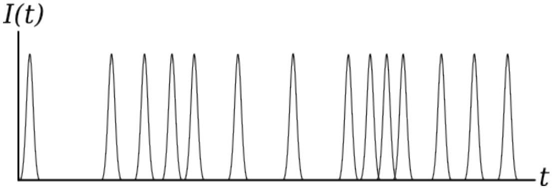


Power spectral density of trading activity (number of trades per 1 min).

$1/f$ noise

- $1/f$ noise is intermediate between white noise, $S(f) \sim 1/f^0$ and Brownian motion $S(f) \sim 1/f^2$.
- Brownian motion can be generated by linear stochastic equation
- The widespread occurring signals and processes with $1/f$ spectrum cannot be understood and modeled in such a way.

Point processes



- The signal of the model consists of pulses or events

$$I(t) = a \sum_k \delta(t - t_k)$$

- Point processes arise in different fields, such as physics, economics, ecology, neurology, seismology, traffic flow, financial systems and the Internet.

Power spectral density

- Stochastic process for the interevent time

$$\tau_{k+1} = \tau_k + a(\tau_k) + b(\tau_k)\varepsilon_k$$

- Approximate expression for power spectral density

$$S(f) = 4\bar{I}^2\bar{\tau}\int_0^\infty \mathrm{d}\tau\, P_\tau(\tau)\,\mathrm{Re}\int_0^\infty \mathrm{d}q\,\exp\left(\mathrm{i}2\pi f[\tau q + a(\tau)q^2/2]\right)$$
$$= \frac{\bar{I}^2\bar{\tau}}{\sqrt{\pi}f}\int_0^\infty P_\tau(\tau)\,\mathrm{Re}\,\frac{\sqrt{x}}{\tau}\mathrm{e}^{-\mathrm{i}x+\mathrm{i}\pi/4}\,\mathrm{erfc}(\sqrt{-\mathrm{i}x})\,\mathrm{d}\tau\,,\qquad x \equiv \frac{\pi f\tau^2}{a(\tau)}$$

- B. Kaulakys, V. Gontis, and M. Alaburda, Phys. Rev. E **71**, 051105 (2005)

Stochastic multiplicative point process

- Stochastic multiplicative process for the interevent time

$$\tau_{k+1} = \tau_k + \gamma \tau_k^{2\mu-1} + \sigma \tau_k^{\mu} \varepsilon_k$$

- Diffusion of the interevent time restricted to the finite interval $[\tau_{\min}, \tau_{\max}]$. Probability density function (PDF) for τ_k

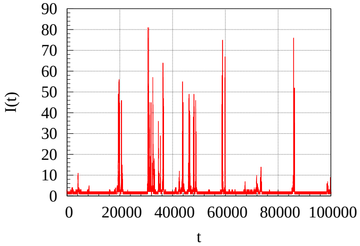
$$P_\tau(\tau_k) = \frac{1+\alpha}{\tau_{\max}^{1+\alpha} - \tau_{\min}^{1+\alpha}} \tau_k^{\alpha}, \qquad \alpha = \frac{2\gamma}{\sigma^2} - 2\mu$$

- Power spectral density in the low frequency limit

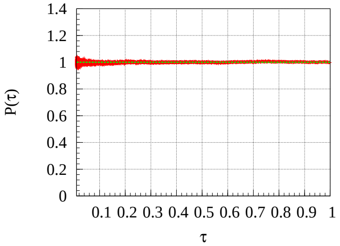
$$S(f) = \frac{(2+\alpha)(\beta-1)\Gamma(\beta-1/2)}{\sqrt{\pi}\alpha(\tau_{\max}^{2+\alpha} - \tau_{\min}^{2+\alpha}) \sin(\pi\beta/2)} \left(\frac{\gamma}{\pi}\right)^{\beta-1} \frac{1}{f^{\beta}},$$

$$\beta = 1 + \frac{\alpha}{3-2\mu}, \qquad 1/2 < \beta < 2$$

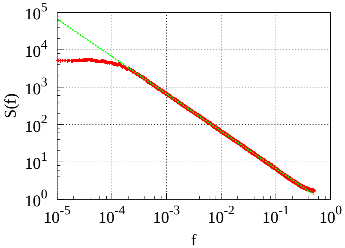
Signal of the point process. Simulated examples



Typical signal



Distribution of τ_k



Power spectral density

Simplest case:

$\gamma = 0$, $\mu = 0$, $\beta = 1$: $1/f$ spectrum.

Transformation to the stochastic differential equation

- Transformation to the Itô stochastic differential equation (SDE) in k-space

$$\mathrm{d}\tau_k = \gamma\tau_k^{2\mu-1}\mathrm{d}k + \sigma\tau_k^{\mu}\mathrm{d}W(k)$$

- Transition from the occurrence number k to the actual time t according to the relation $\mathrm{d}t = \tau_k \mathrm{d}k$ yields

$$\mathrm{d}\tau = \gamma\tau^{2\mu-2}\mathrm{d}t + \sigma\tau^{\mu-\frac{1}{2}}\mathrm{d}W$$

- Averaged over time interval τ_k intensity of the signal $x = \frac{1}{\tau}$. Transformation of the variable from τ to x yields

$$\mathrm{d}x = \sigma^2(\eta - \lambda/2)x^{2\eta-1}\mathrm{d}t + \sigma x^{\eta}\mathrm{d}W \tag{1}$$

where

$$\eta = 5/2 - \mu\,, \qquad \lambda = 2(\eta - 1 + \gamma/\sigma^2)$$

Stochastic differential equation for the signal

Equation (1) generates signals with the power-law distribution of the signal intensity and power spectral density

$$P(x) \sim x^{-\lambda}\,, \qquad S(f) \sim f^{-\beta}\,, \qquad \beta = 1 - \frac{\lambda - 3}{2\eta - 2}$$

Restricted diffusion

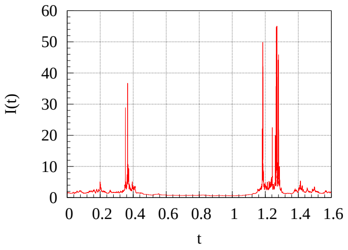
Exponentially restricted diffusion with the distribution densities

$$P(x) \sim \frac{1}{x^{\lambda}} \exp\left\{ -\left(\frac{x_{\min}}{x}\right)^m - \left(\frac{x}{x_{\max}}\right)^m \right\}$$

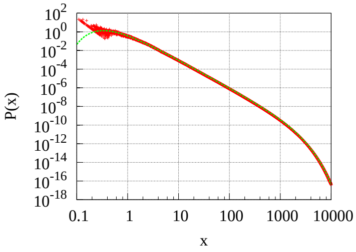
Is generated by the stochastic differential equation

$$\mathrm{d}x = \sigma^2 \left[\eta - \frac{\lambda}{2} + \frac{m}{2} \left(\frac{x_{\min}^m}{x^m} - \frac{x^m}{x_{\max}^m} \right) \right] x^{2\eta-1} \mathrm{d}t + \sigma x^{\eta} \mathrm{d}W$$

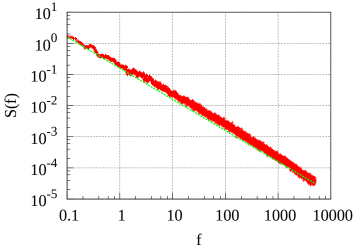
Numerical simulation



Typical signal



Distribution of x



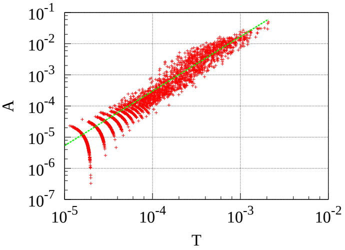
Power spectral density

Used parameters: $\lambda = 3$, $\eta = 5/2$, $x_{\mathrm{min}} = 1.0$, $x_{\mathrm{max}} = 10^3$.
$1/f$ spectrum.

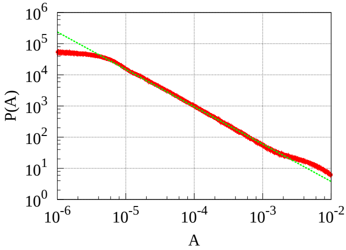
Signals consisting of bursts

- Numerical analysis indicates that solutions of SDE (1) have a second structure composed of peaks, bursts, clusters of events.
- Bursts are characterized by power-law distributions of burst size, burst duration, and interburst time.
- Proposed nonlinear SDE may simulate
  - avalanches in self-organized critical (SOC) models
  - extreme event return times in long memory processes
- B. Kaulakys and M. Alaburda, J. Stat. Mech. P02051 (2009).

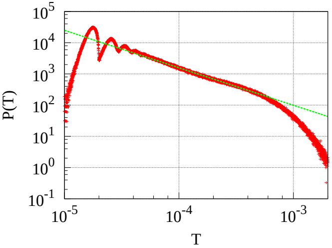
Signals consisting of bursts



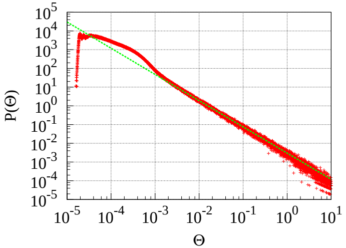
Area of burst $A \sim T^2$



Distribution of burst area $P(A) \sim A^{-1.2}$



burst duration: $P(T) \sim T^{-1.2}$



interburst time: $P(\Theta) \sim \Theta^{-1.4}$

Possibility of changing SDE

- $1/f^{\beta}$ power spectral density is determined mainly by power-law behavior of the coefficients of SDE (1) at big x. Changing the coefficients at small x, spectrum retains power-law behavior.
- PDF from associated Fokker-Planck equation has power-law dependence on x.
- Therefore, SDE (1) can be naturally connected with generalized canonical distributions of nonextensive statistical mechanics — q-exponentials.

q-exponential is defined as

$$\exp_q(x) \equiv (1 + (1-q)x)^{\frac{1}{1-q}}$$

q-exponential distribution

Stochastic differential equation

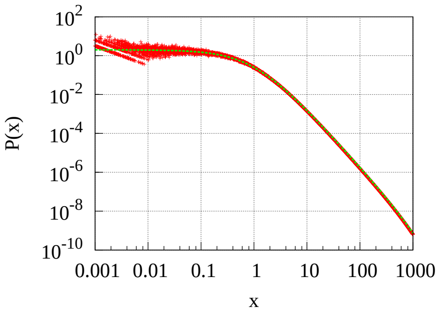
$$\mathrm{d}x = \sigma^2(\eta - \lambda/2)(x + x_0)^{2\eta - 1}\mathrm{d}t + \sigma(x + x_0)^{\eta}\mathrm{d}W$$

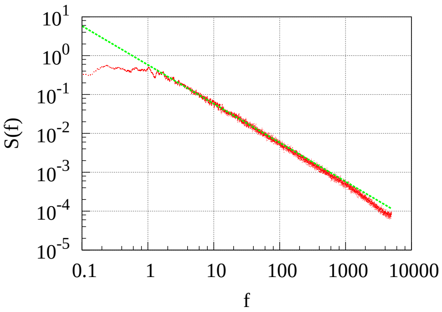
- is linear for small $x \ll x_0$
- restrict divergence of power-law distribution of x at $x = 0$
- generates stochastic processes with power spectral density $S(f) \sim 1/f^{\beta}$
- probability density function in the form of q-exponential

$$P(x) = \frac{\lambda - 1}{x_0}\left(\frac{x_0}{x + x_0}\right)^{\lambda} = \frac{\lambda - 1}{x_0}\exp_q(-\lambda x/x_0)$$

$$q = 1 + 1/\lambda$$

Numerical simulation





Numerically simulated distribution density and power spectral density ($\lambda = 3$, $\eta = 5/2$).

q-Gaussian distribution

Stochastic differential equation

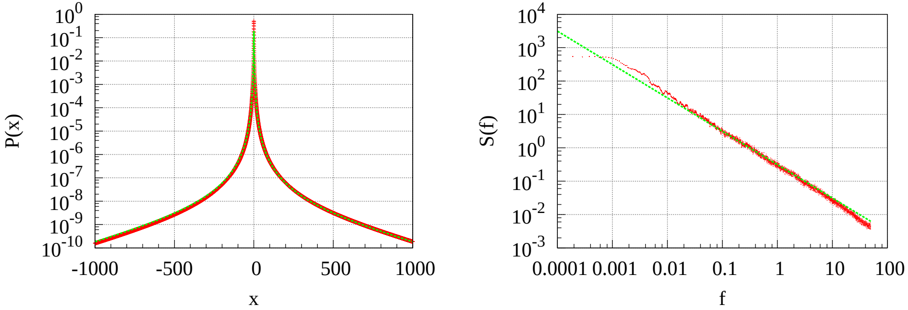
$$\mathrm{d}x = \sigma^2(\eta - \lambda/2)(x_0^2 + x^2)^{\eta-1} x \mathrm{d}t + \sigma(x_0^2 + x^2)^{\eta/2} \mathrm{d}W$$

- allows negative values of x
- for small $x \ll x_0$ is linear, becomes ordinary Brownian motion with relaxation
- generates stochastic processes with power spectral density $S(f) \sim 1/f^{\beta}$
- probability density function in the form of q-Gaussian

$$P(x) = \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi} x_0 \Gamma\left(\frac{\lambda-1}{2}\right)} \left(\frac{x_0^2}{x_0^2 + x^2}\right)^{\frac{\lambda}{2}} = \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi} x_0 \Gamma\left(\frac{\lambda-1}{2}\right)} \exp_q\left(-\lambda \frac{x^2}{2x_0^2}\right)$$

$$q = 1 + 2/\lambda$$

Numerical simulation



Numerically simulated distribution density and power spectral density ($\lambda = 3$, $\eta = 5/2$).

Slowly changing mean

Another connection with q-exponential:

- Stochastic process with slowly fluctuating mean driven by the multiplicative stochastic differential equation.
- Superstatistical approach
- At small frequencies the spectrum is determined by the driving SDE.

q-exponential distribution

Conditional probability

$$\varphi(x|\bar{x}) = \bar{x}^{-1}\exp(-x/\bar{x})\,.$$

The mean $\bar{x}$ changes according to the equation

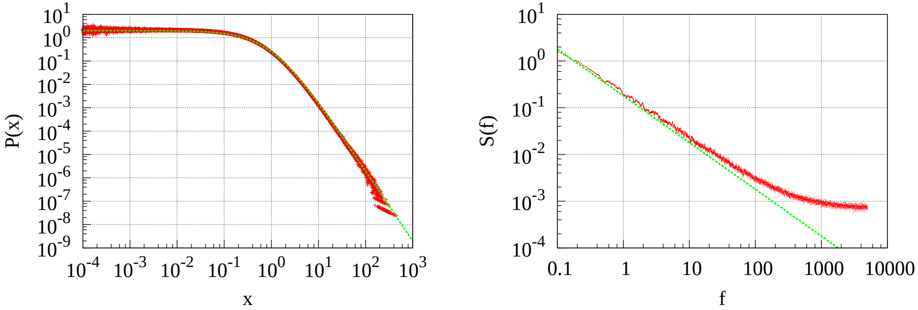
$$\mathrm{d}\bar{x} = \sigma^2\left[\eta - \frac{\lambda}{2} + \frac{1}{2}\frac{x_0}{\bar{x}}\right]\bar{x}^{2\eta-1}\mathrm{d}t + \sigma\bar{x}^{\eta}\mathrm{d}W$$

The stationary PDF from associated Fokker-Planck equation is

$$P(\bar{x}) = \left(x_0\Gamma(\lambda-1)\right)^{-1}(x_0/\bar{x})^{\lambda}\exp(-x_0/\bar{x})$$

Then PDF of x is q-exponential

$$\begin{aligned} P(x) &= \textstyle\int_0^\infty \varphi(x|\bar{x})P(\bar{x})\,\mathrm{d}\bar{x} = \frac{\lambda-1}{x_0}\left(\frac{x_0}{x+x_0}\right)^{\lambda} \\ &= \textstyle\frac{\lambda-1}{x_0}\exp_q(-\lambda x/x_0)\,, \qquad q = 1 + 1/\lambda \end{aligned}$$



Numerically simulated distribution density and power spectral density ($\lambda = 3$, $\eta = 5/2$, $x_0 = 1.0$).

q-Gaussian distribution

Conditional probability

$$\varphi(x|\bar{x}) = 2\left(\sqrt{\pi}\bar{x}\right)^{-1}\exp(-x^2/\bar{x}^2)\,,\qquad x \geqslant 0$$

The mean is proportional to $\bar{x}$ which changes according to the equation

$$\mathrm{d}\bar{x} = \sigma^2\left[\eta - \frac{\lambda}{2} + \frac{x_0^2}{\bar{x}^2}\right]\bar{x}^{2\eta-1}\mathrm{d}t + \sigma\bar{x}^{\eta}\mathrm{d}W$$

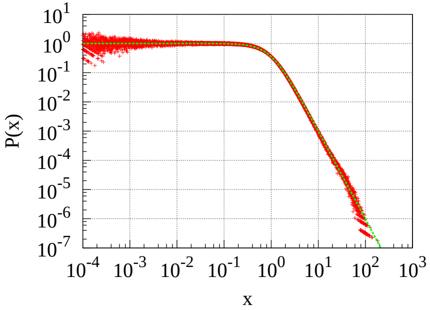
The stationary PDF from associated Fokker-Planck equation is

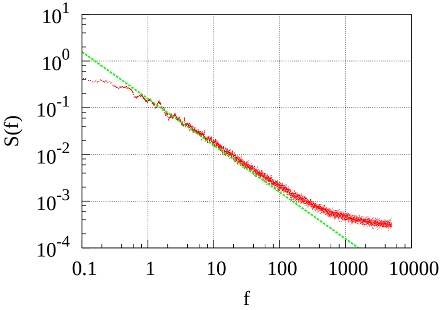
$$P(\bar{x}) = \frac{2}{x_0\Gamma\left(\frac{\lambda-1}{2}\right)}(x_0/\bar{x})^{\lambda}\exp(-x_0^2/\bar{x}^2)$$

Then PDF of x is q-Gaussian

$$P(x) = \frac{2\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi}x_0\Gamma\left(\frac{\lambda-1}{2}\right)}\left(\frac{x_0^2}{x_0^2+x^2}\right)^{\frac{\lambda}{2}} = \frac{2\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi}x_0\Gamma\left(\frac{\lambda-1}{2}\right)}\exp_q\left(-\lambda\frac{x^2}{2x_0^2}\right)$$

$$q = 1 + 2/\lambda$$





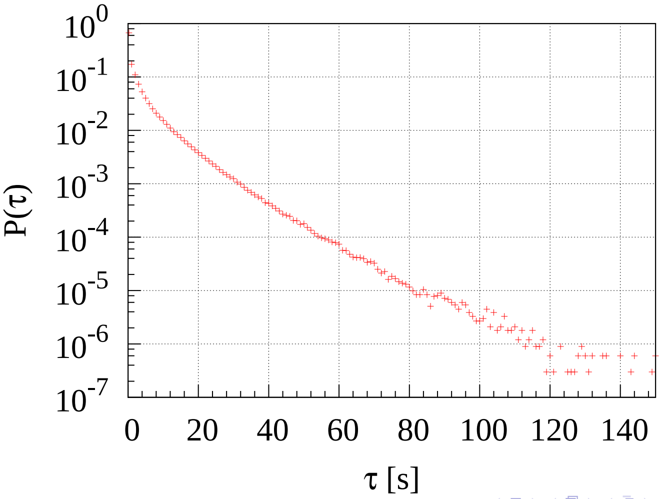
Numerically simulated distribution density and power spectral density ($\lambda = 3$, $\eta = 5/2$, $x_0 = 1.0$).

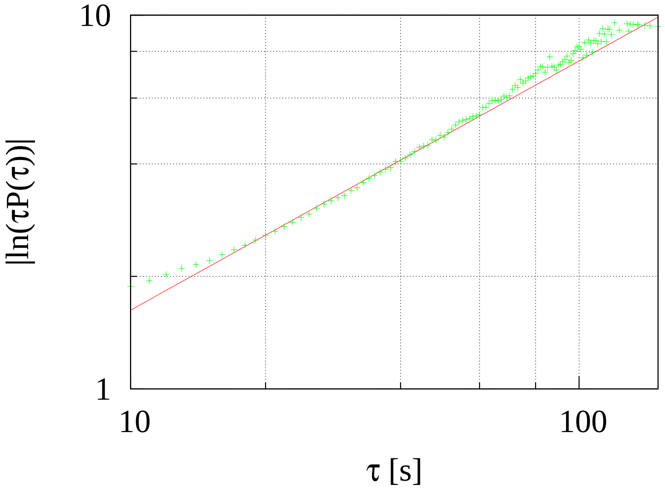
Application to the description of trading

- Time series of financial data exhibit highly nontrivial statistical properties. Many of these properties appear to be universal.
- Trading activity, trading volume, and volatility are stochastic variables with the long-range correlation. The autocorrelation of the volatility decays only slowly as a power law
- PDF of return and trading activity have fat tails exhibiting power-law decay.
- Proposed equations can exhibit both power-law PDF and power-law spectrum.

Intertrade time

The distribution of intertrade time τ at big intertrade times is close to stretched exponential.





The slope of the line is $2/3$

Modeling of trading sequence

Trade sequence — Poissonian like point process with slowly fluctuating mean intertrade time. Each individual intertrade duration τ is distributed according to exponential distribution

$$\varphi(\tau|n) = n\exp(-n\tau)$$

The distribution of trading activity:

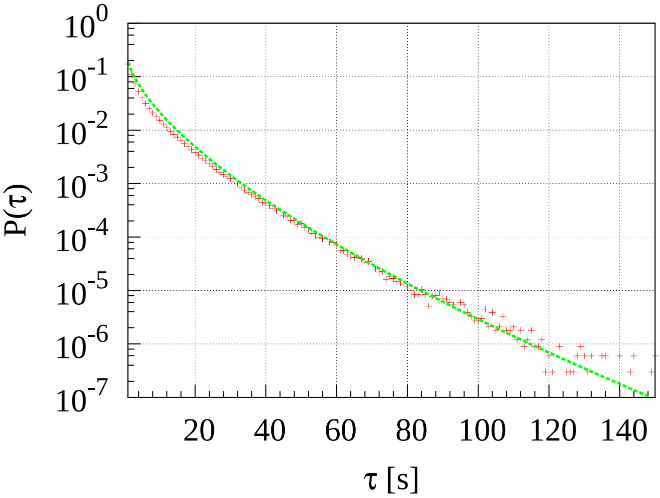
$$P_n(n) = \frac{2}{n_0\Gamma\left(\frac{\lambda-1}{2}\right)}(n_0/n)^{\lambda}\exp(-n_0^2/n^2) \qquad (2)$$

The distribution of intertrade durations

$$P(\tau) = \int_0^{\infty}\frac{n}{\langle n\rangle}\varphi(\tau|n)P_n(n)\,\mathrm{d}n \underset{\tau\to\infty}{\longrightarrow} \left(\frac{\tau}{\tau_0}\right)^{\frac{\lambda-4}{3}}\exp\left(-3\left(\frac{\tau}{2\tau_0}\right)^{\frac{2}{3}}\right)$$

Here $\langle n\rangle = \int_0^{\infty} nP_n(n)\mathrm{d}n$ is average trading activity and $\tau_0 = 1/n_0$.

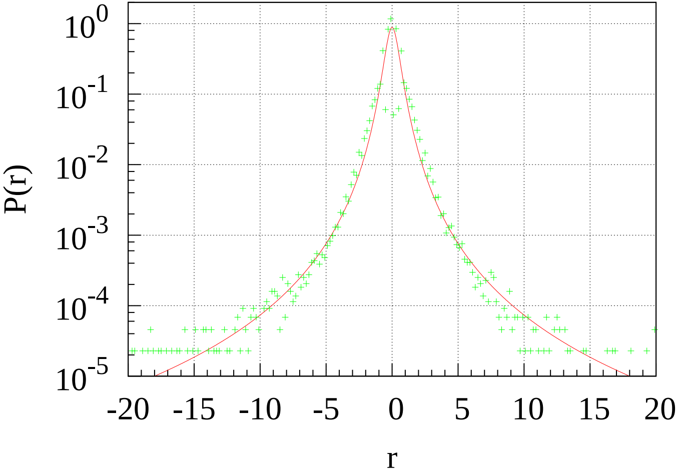
Comparison with empirical data



Used parameters: $\lambda = 3.4$, $\tau_0 = 7.3\,\mathrm{s}$.

Trading return

The distribution of normalized return r per 1 min is close to q-Gaussian.



Modeling of return

Standard deviation of return is proportional to the trading activity.

$$\varphi(r|n) = \frac{1}{\sqrt{\pi}an} \exp\left(-\frac{r^2}{a^2n^2}\right)$$

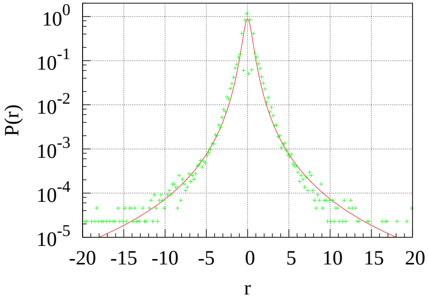
a is the coefficient of proportionality.
Using the same distribution of trading activity (2) as in modeling the intertrade duration, we get the distribution of return

$$P(r) = \int_0^\infty \varphi(r|n)P_n(n)\,\mathrm{d}n = \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi}r_0\Gamma\left(\frac{\lambda-1}{2}\right)}\left(\frac{r_0^2}{r_0^2+r^2}\right)^{\frac{\lambda}{2}}$$
$$= \frac{2\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi}r_0\Gamma\left(\frac{\lambda-1}{2}\right)} \exp_q\left(-\lambda\frac{r^2}{2r_0^2}\right), \qquad q = 1 + 2/\lambda$$

Here $r_0 = an_0$.

Comparison with empirical data

Normalized return: $r = (r - \langle\tilde{r}\rangle)/\sqrt{\langle(\tilde{r} - \langle\tilde{r}\rangle)^2\rangle}$.
For normalized return $r_0 = \sqrt{\lambda - 3}$



The same parameters as for intertrade duration: $\lambda = 3.4$ ($q = 1.59$).

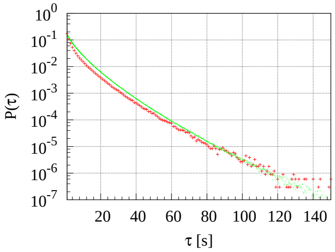
SDE for trading activity

- In order to reproduce the empirical data for both the PDF and the spectrum of the trading activity in the financial markets, modification of the equation is proposed

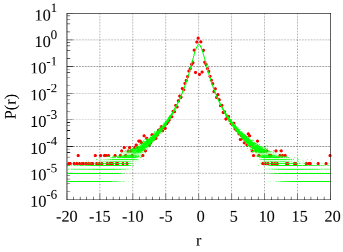
$$\mathrm{d}n = \sigma^2 \left[5/2 - \lambda/2 + \frac{n_0^2}{n^2} \right] \frac{n^4}{(n\epsilon + 1)^2} \mathrm{d}t + \frac{\sigma n^{5/2}}{(n\epsilon + 1)} \mathrm{d}W$$

  ϵ defines crossover between two areas of n diffusion.
- Parameters: $\lambda = 4.28$, $(q = 1.47)$, $\sigma/\tau_0 = 0.005$, $\epsilon/\tau_0 = 0.03$.
- V. Gontis, B. Kaulakys, J. Ruseckas, Physica A **387**, 3891 (2008).
- Problem of equation: unnaturally high spikes of activity.

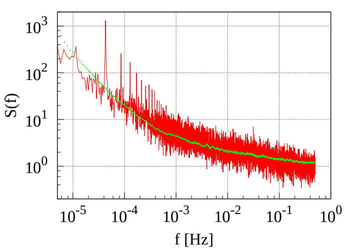
Spectrum and distribution



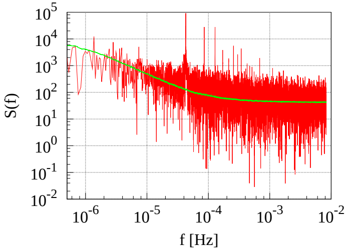
PDF of a sequence of trades



PDF of normalized return



Spectrum of a sequence of trades



Spectrum of absolute normalized return

Summary

- Long-range memory processes with the power-law distributions of the signal intensity and power spectral density can be modeled by nonlinear stochastic differential equations.
- Generalized canonical distributions of nonextensive statistical mechanics can be obtained from proposed nonlinear SDE.
- PDF of intertrade time and PDF of return can be obtained using superstatistical approach from trading activity, whereas trading activity may be modeled by proposed nonlinear SDE. Such approach readily gives power-law spectrum of intertrade time and absolute return.

Thank you for your attention!