Nonlinear stochastic differential equations and $1/f$ noise

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Outline

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1. Introduction: $1/f$ noise
2. Stochastic differential equations giving $1/f$ noise
3. Some models resulting in proposed SDE
   - Point processes
   - Simple model of herding behavior
4. Summary

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What is $1/f$ noise?

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a type of noise whose power spectral density $S(f)$ behaves like

$$S(f) \sim 1/f^{\beta}, \qquad \beta \text{ is close to } 1$$

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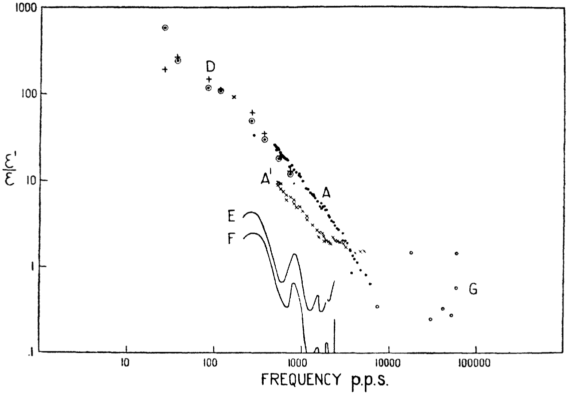


Fig. 6. Frequency variation for tube No. 2, coated filament; same data as in Fig. 4 plotted to a frequency scale; curves E and F give Hartmann's results for 2 m-a. and 20 m-a.; points G were obtained with less steady measuring circuit.

First observed (in 1925) by Johnson in vacuum tubes.

1/*f* noise

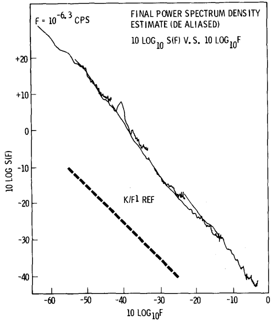


FIG. 8. Final power spectrum density estimate (dealiased).

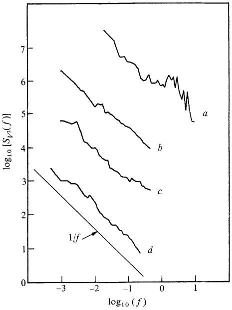
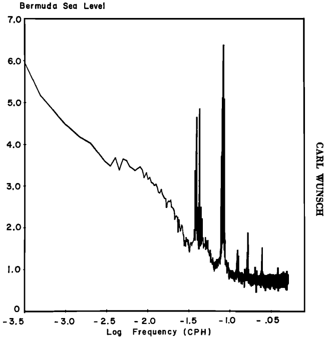


Fig. 2 Loudness fluctuation spectra, $S_{V^2}(f)$ against f for: *a*, Scott Joplin Piano Rags; *b*, classical radio station; *c*, rock station; *d*, news and talk station.



Fluctuations of signals exhibiting $1/f$ behavior of the power spectral density at low frequencies have been observed in a **wide variety** of physical, geophysical, biological, financial, traffic, Internet, astrophysical and other systems.

Many mathematical models:

- Superposition of relaxation processes

$$S(f)=\int_{\gamma_1}^{\gamma_2}\frac{N}{\gamma^2+\omega^2}\,\mathrm{d}\gamma\approx\frac{\pi N}{2\omega}\,,\qquad \gamma_1\ll\omega\ll\gamma_2$$

- Dynamical systems at the edge of chaos

$$x_{n+1}=x_n+x_n^z \mod 1$$

- Alternating fractal renewal process

- Self-Organized Criticallity

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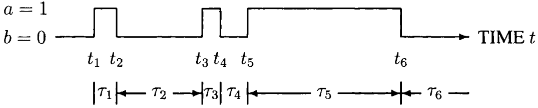

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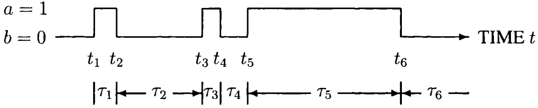

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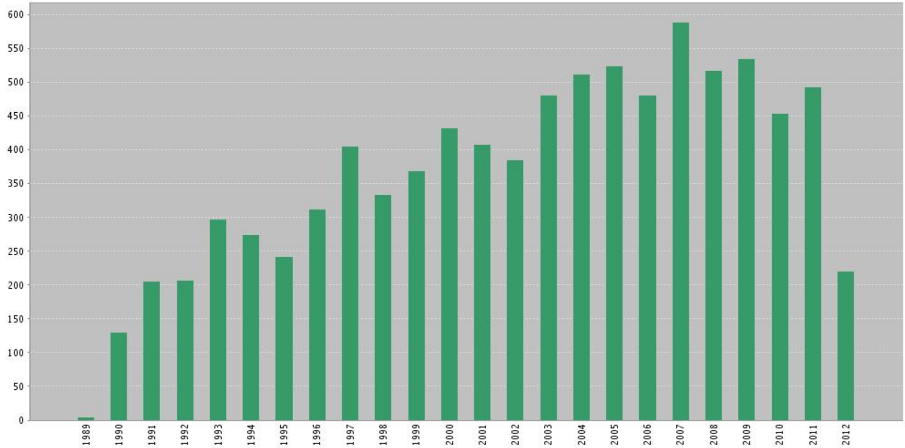
A bibliography on $1/f$ noise is vast



Published items in each year. Topic: $1/f$ noise, $1/f$ fluctuations, flicker noise, pink noise (Web of Science)

$1/f$ noise

- $1/f$ noise is intermediate between white noise, $S(f) \sim 1/f^0$ and Brownian motion $S(f) \sim 1/f^2$
- In contrast to the Brownian motion generated by the linear stochastic equations, the signals and processes with $1/f$ spectrum **cannot** be understood and modeled in such a way.

Goal

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Notes about $1/f$ noise

- Often $1/f$ noise is defined by a long-memory process, characterized by $S(f) \sim 1/f^{\beta}$ as $f \to 0$.
- A **pure** $1/f$ power spectrum is **physically impossible** because the total power would be infinity.
- We search for a model where the spectrum of signal has $1/f^{\beta}$ behavior only in some **intermediate** region of frequencies, $f_{\min} \ll f \ll f_{\max}$, whereas for small frequencies $f \ll f_{\min}$ the spectrum is bounded.

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Heuristic derivation of SDE

- If $S(f) \sim f^{-\beta}$ then power spectral density has a scaling property

$$S(af) = a^{-\beta} S(f)$$

- Wiener-Khintchine theorem

$$C(t) = \int_0^{+\infty} S(f) \cos(2\pi ft)\, \mathrm{d}f$$

- Autocorrelation function $C(t)$ has scaling property

$$C(at) \sim a^{\beta-1} C(t)$$

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$$C(t) = \int \mathrm{d}x \int \mathrm{d}x' \, xx' P_0(x) P_x(x', t|x, 0)$$

- $P_0(x)$ is the steady state PDF
- $P_x(x', t|x, 0)$ is the transition probability
- The transition probability can be obtained from the solution of the **Fokker-Planck equation** with the initial condition $P_x(x', 0|x, 0) = \delta(x' - x)$.

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- Let us assume that
  - Steady state PDF has power-law form

$$P_0(x) \sim x^{-\nu}$$

  - Trasnsition probability has a scaling property

$$P(ax', t|ax, 0) = a^{-1} P(x', a^{2(\eta-1)} t|x, 0)$$

- Then the autocorrelation function will have the required scaling with

$$\beta = 1 + \frac{\nu - 3}{2(\eta - 1)}$$

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- SDE should contain only powers of x
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Proposed SDE

$$\mathrm{d}x = \sigma^2(\eta - \nu/2)x^{2\eta-1}\mathrm{d}t + \sigma x^{\eta}\mathrm{d}W_t$$

B. Kaulakys and J. Ruseckas, Phys. Rev. E **70**, 020101(R) (2004).

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Transformation property

Introducing

$$z = x^{\alpha}$$

we get the equation **of the same type**

$$\mathrm{d}z = \sigma'^2(\eta' - \nu'/2)z^{2\eta'-1}\mathrm{d}t + \sigma' z^{\eta'}\mathrm{d}W_t$$

only with different parameters

$$\sigma' = \alpha\sigma\,, \quad \eta' = (\eta - 1)/\alpha + 1\,, \quad \nu' = (\nu - 1)/\alpha + 1$$

Restriction of diffusion

- Because of the divergence of the power-law distribution and the requirement of the stationarity of the process, the SDE should be analyzed together with the appropriate **restrictions** of the diffusion in some finite interval.
- When diffusion is restricted, **scaling properties are only approximate**, but $1/f$ spectrum remains in a wide interval of frequencies.

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Restriction of diffusion

Possible forms of restriction:

- Reflective boundary conditions at $x = x_{\min}$ and $x = x_{\max}$
- Exponential restriction of the diffusion

$$\mathrm{d}x = \sigma^2 \left(\eta - \frac{\nu}{2} + \frac{m}{2}\left(\frac{x_{\min}}{x}\right)^m - \frac{m}{2}\left(\frac{x}{x_{\max}}\right)^m \right) x^{2\eta-1}\mathrm{d}t + \sigma x^{\eta}\mathrm{d}W_t$$

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$$P_0(x) \sim \exp_{1+1/\nu}(-\nu x/x_0)$$

  Reflective boundary condition at $x = 0$

- q-Gaussian steady-state PDF

$$\mathrm{d}x = \sigma^2(\eta - \nu/2)(x^2 + x_0^2)^{\eta-1}x\mathrm{d}t + \sigma(x^2 + x_0^2)^{\eta/2}\mathrm{d}W_t$$

$$P_0(x) \sim \exp_{1+2/\nu}(-\nu x^2/2x_0^2)$$

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J. Ruseckas and B. Kaulakys, Phys. Rev. E **84**, 051125 (2011).

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Connection with other equations

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- $\eta = 0$ and $\sigma = 1$ corresponds to the **Bessel process**

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- $\eta = 1/2$, $\sigma = 2$ corresponds to the **squared Bessel process**

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Connection with other equations

- SDE with exponential restriction with $\eta = 1/2$, $x_{\min} = 0$ and $m = 1$ gives **Cox-Ingersoll-Ross (CIR) process**

$$\mathrm{d}x = k(\theta - x)\mathrm{d}t + \sigma\sqrt{x}\,\mathrm{d}W_t$$

where $k = \sigma^2/2x_{\max}$, $\theta = x_{\max}(1 - \nu)$

- When $\nu = 2\eta$, $x_{\max} = \infty$ and $m = 2\eta - 2$ then we get the **Constant Elasticity of Variance (CEV) process**

$$\mathrm{d}x = \mu x \mathrm{d}t + \sigma x^{\eta} \mathrm{d}W_t$$

where $\mu = \sigma^2(\eta - 1)x_{\min}^{2(\eta-1)}$

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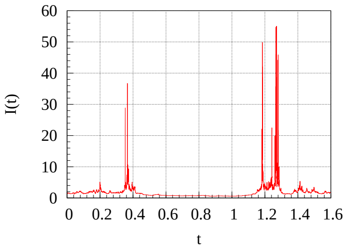
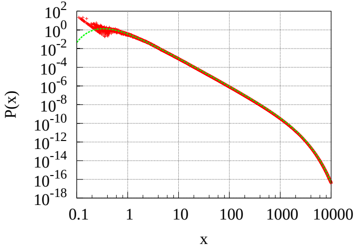
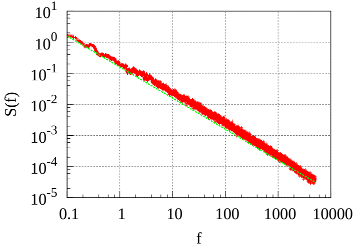
Numerical example



Typical signal



Distribution of x



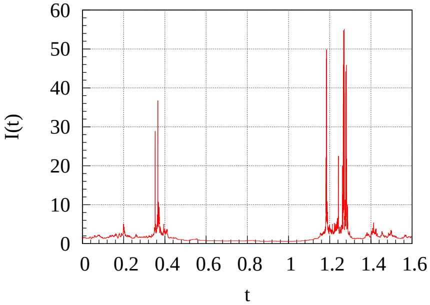
Power spectral density

$$\mathrm{d}x = \left(1 + \frac{x_{\min}}{2x} - \frac{x}{2x_{\max}}\right)x^4 \mathrm{d}t + x^{\frac{5}{2}}\mathrm{d}W_t$$

$\nu = 3$, $\eta = \frac{5}{2}$, $x_{\min} = 1$, $x_{\max} = 10^3$.

$1/f$ spectrum.

Intermittent behavior of solutions



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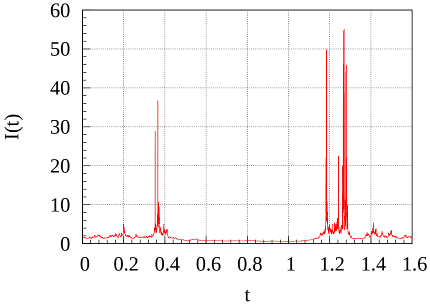

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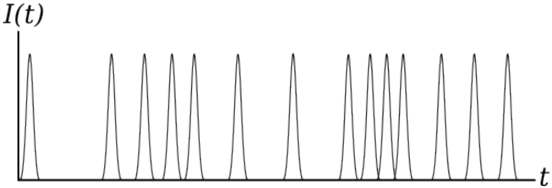
Outline

Point processes

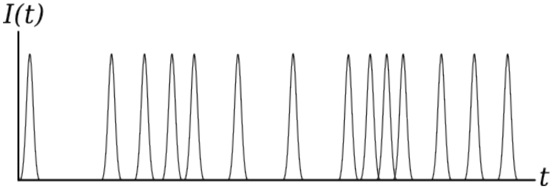


- The signal of the model consists of pulses or events

$$I(t) = a \sum_{k} \delta(t - t_k)$$

- Point processes arise in different fields such as physics, economics, ecology, neurology, seismology, traffic flow, financial systems and the Internet.

Point processes



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How to obtain equation for inter-event time $\tau_k = t_k - t_{k-1}$:

- Transform the equation from the variable x to $\tau = 1/x$
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$$\mathrm{d}x = \sigma^2 x^4 \mathrm{d}t + \sigma x^{5/2} \mathrm{d}W$$

leads to

$$\tau_{k+1} = \tau_k + \sigma\varepsilon_k$$

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where $\mu = 5/2 - \eta$, $\gamma = \sigma^2(1 - \eta + \nu/2)$.
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- fixed number N of agents
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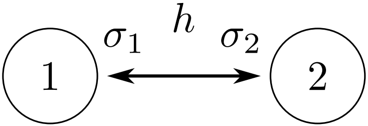
$$p(n \to n+1) = (N-n)(\sigma_1 + hn)$$
$$p(n \to n-1) = n(\sigma_2 + h(N-n))$$



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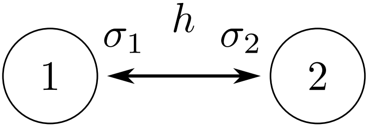

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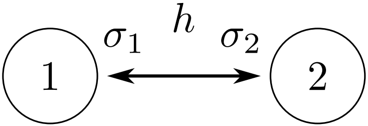

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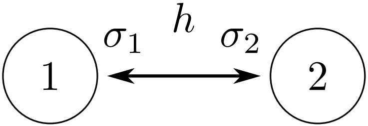

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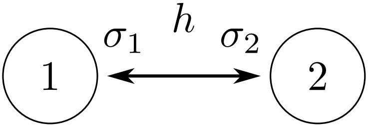

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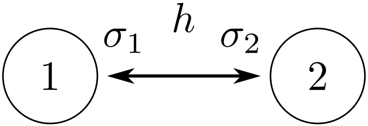

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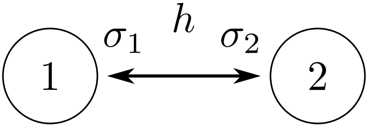

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- Ratio of the number of agents in the state 2 to the number of agents in the state 1:

$$y = \frac{N - n}{n}$$

- For large N we can represent the dynamics by SDE

$$\mathrm{d}y = [(2h - \sigma_1)y + \sigma_2](1 + y)\mathrm{d}t + \sqrt{2hy}(1 + y)\mathrm{d}W$$

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If $\sigma_1 = 2h$, we obtain $1/f$ spectrum

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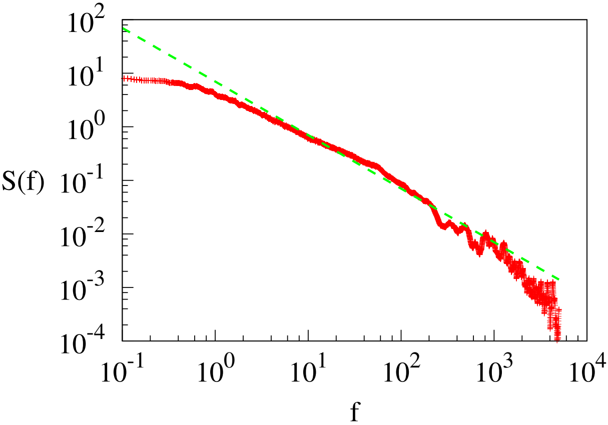

Power spectral density of the ratio of the numbers of agents.
$N = 10\,000$

Summary

- We obtain a class of nonlinear SDEs giving the power-law behavior of the power spectral density in any desirably wide range of frequencies
- and power-law steady state distribution of the signal intensity.
- In special cases we obtain other well-known SDEs.
- One of the reasons for the appearance of the $1/f$ spectrum are scaling properties of the SDE.
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