Modeling point processes by the stochastic differential equation

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Outline

$1/f$ noise

- Brownian motion without correlations between increments results in $1/f^2$ and Lorentzian power spectral densities
- The widespread occurring signals and processes with $1/f$ spectrum cannot be understood and modeled in such a way
- $1/f$ noise is a type of noise whose power spectral density $S(f)$ as a function of the frequency f behaves like $S(f) \sim 1/f^{\beta}$ where the exponent β is close to 1.
- Fluctuations of signals exhibiting $1/f$ behavior of the power spectral density at low frequencies have been observed in a wide variety of physical, geophysical, biological, financial, traffic, the Internet, astrophysical and other systems.

$1/f$ noise

- $1/f$ noise is intermediate between white noise, $S(f) \sim 1/f^0$ and Brownian motion $S(f) \sim 1/f^2$
- In contrast to the Brownian motion generated by the linear stochastic equation, simple systems of differential equations, even linear stochastic equations, generating signals with $1/f$ noise are not known.
- These results make the problem of the omnipresent $1/f$ noise one of the oldest puzzles in contemporary physics.

Point processes

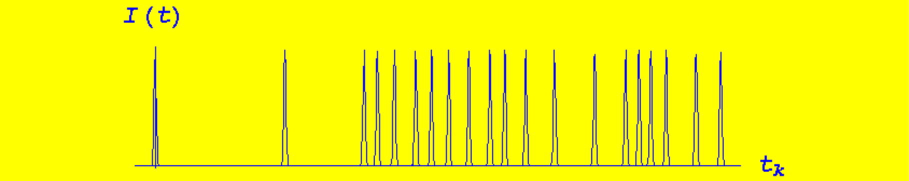
- The signal of the model consists of pulses or events

$$I(t) = \sum_k A_k(t - t_k)$$

- In a low frequency region and for long-range correlations we can restrict analysis to the noise originated from the correlations between the occurrence times t_k.
- Therefore, we can simplify the signal to the point process

$$I(t) = \sum_k \delta(t - t_k)$$

Point processes



- The point process is primarily and basically defined by the occurrence times $t_1, t_2, \ldots t_k \ldots$
- Point processes arise in different fields, such as physics, economics, ecology, neurology, seismology, traffic flow, financial systems and the Internet.

Process for interevent time

- Interevent time
$$\tau_k = t_{k+1} - t_k$$
- We consider stochastic multiplicative process for the interevent time
$$\tau_{k+1} = \tau_k + \gamma\tau_k^{2\mu-1} + \sigma\tau_k^{\mu}\varepsilon_k$$
- The process may generate signals with the power-law distributions of the signal intensity and $1/f^{\beta}$ noise.

Multiplicative point process

- We restrict the diffusion of the interevent time to the finite interval $[\tau_{\mathrm{min}}, \tau_{\mathrm{max}}]$. Probability density function for τ_k

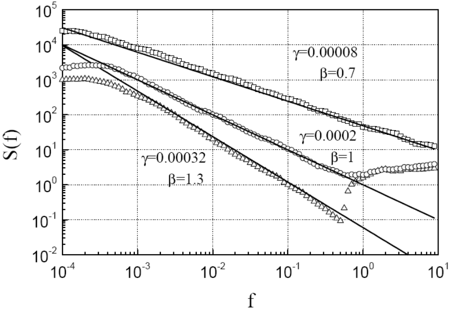
$$P_k(\tau_k) = \frac{1+\alpha}{\tau_{\mathrm{max}}^{1+\alpha} - \tau_{\mathrm{min}}^{1+\alpha}} \tau_k^{\alpha}\,, \qquad \alpha = \frac{2\gamma}{\sigma^2} - 2\mu$$

- Power spectral density in the low frequency limit

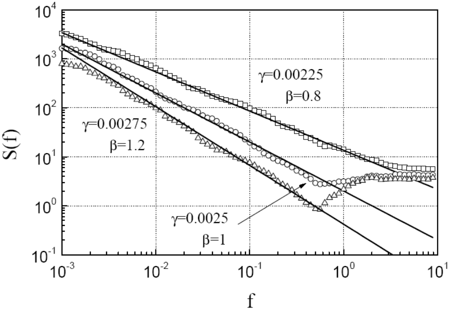
$$S(f) = \frac{(2+\alpha)(\beta-1)\Gamma(\beta-1/2)}{\sqrt{\pi}\alpha(\tau_{\mathrm{max}}^{2+\alpha} - \tau_{\mathrm{min}}^{2+\alpha})\sin(\pi\beta/2)} \left(\frac{\gamma}{\pi}\right)^{\beta-1} \frac{1}{f^{\beta}},$$

$$\beta = 1 + \frac{\alpha}{3-2\mu}, \qquad \frac{1}{2} < \beta < 2$$

Numerical analysis



$\mu = \frac{1}{2}, \sigma = 0.02$



$\mu = 1, \sigma = 0.02$

Transformation to the stochastic differential equation

- Transformation to the Itô stochastic differential equation in k-space

$$\mathrm{d}\tau_k = \gamma\tau_k^{2\mu-1}\mathrm{d}k + \sigma\tau_k^{\mu}\mathrm{d}W(k)$$

- Transition from the occurence number k to the actual time t according to the relation

$$\mathrm{d}t = \tau_k \mathrm{d}k$$

yields

$$\mathrm{d}\tau = \gamma\tau^{2\mu-2}\mathrm{d}t + \sigma\tau^{\mu-\frac{1}{2}}\mathrm{d}W$$

Stochastic differential equation for the signal

Averaged over time interval τ_k intensity of the signal

$$n = \frac{1}{\tau}$$

Transformation of the variable from τ to n yields

$$\mathrm{d}n = \sigma^2(1-\gamma_\sigma)n^{2\eta-1}\mathrm{d}t + \sigma n^\eta \mathrm{d}W \tag{1}$$

where

$$\eta = \frac{5}{2} - \mu, \quad \gamma_\sigma = \frac{\gamma}{\sigma^2}\,.$$

Stochastic differential equation for the signal

Equation (1) generates signals with the power-law distribution of the signal intensity

$$P(n) \sim \frac{1}{n^{\lambda}}, \qquad \lambda = 2(\eta - 1 + \gamma_{\sigma})$$

and noise

$$S(f) \sim \frac{1}{f^{\beta}}, \qquad \beta = 2 - \frac{3 - 2\gamma_{\sigma}}{2\eta - 2}$$

Restricted diffusion

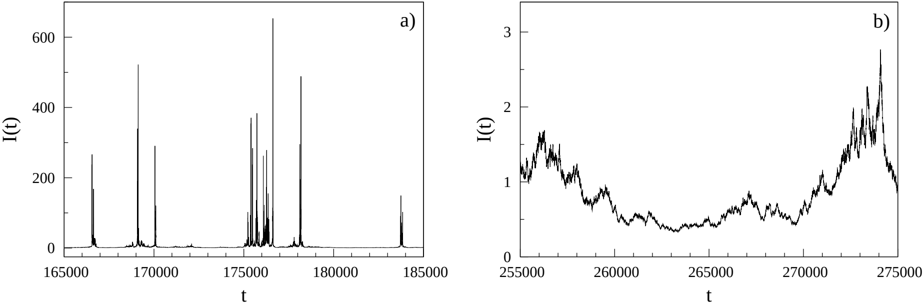
Exponentially restricted diffusion with the distribution densities

$$P(n) \sim \frac{1}{n^{\lambda}} \exp\left\{ -\left(\frac{n_{\min}}{n}\right)^{m} - \left(\frac{n}{n_{\max}}\right)^{m} \right\}$$

Is generated by the stochastic differential equation

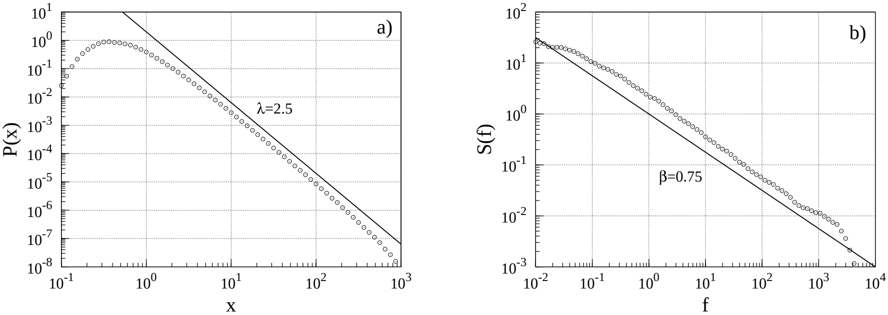
$$\mathrm{d}n = \sigma^2 \left[1 - \gamma_\sigma + \frac{m}{2} \left(\frac{n_{\min}^m}{n^m} - \frac{n^m}{n_{\max}^m} \right) \right] n^{2\eta-1} \mathrm{d}t + \sigma n^{\eta} \mathrm{d}W$$

Numerical simulation



Typical examples of the solutions: a) with the parameters $\gamma_\sigma = 0.25$, $\eta = 2$ and b) with the parameters $\gamma_\sigma = 1.2$, $\eta = 1.5$

Numerical simulation



Numerically simulated distribution density a) and power spectral density b)

Stochastic model of trading activity

- In order to reproduce the empirical data for both the PDF and the spectrum of the trading activity in the financial markets, modification of the equation (1) is proposed

$$\mathrm{d}n = \sigma^2 \left[(1-\gamma_\sigma) + \frac{m}{2}\left(\frac{n_0}{n}\right)^m \right] \frac{n^4}{(n\epsilon+1)^2}\mathrm{d}t + \frac{\sigma n^{5/2}}{(n\epsilon+1)}\mathrm{d}W$$

  where ϵ defines crossover between two areas of n diffusion.

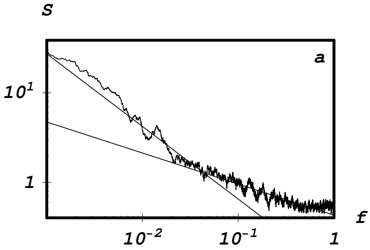
- The corresponding iterative equation for τ_k is

$$\tau_{k+1} = \tau_k + \left[\gamma - \frac{m}{2}\sigma^2 \left(\frac{\tau_k}{\tau_m}\right)^m \right] \frac{\tau_k}{(\epsilon+\tau_k)^2} + \sigma \frac{\tau_k}{\epsilon+\tau_k}\varepsilon_k$$

- Continuous equation in the actual time

$$\mathrm{d}\tau = \left[\gamma - \frac{m}{2}\sigma^2 \left(\frac{\tau}{\tau_m}\right)^m \right] \frac{1}{(\epsilon+\tau)^2}\mathrm{d}t + \sigma \frac{\sqrt{\tau}}{\epsilon+\tau}\mathrm{d}W$$

Stochastic model of trading activity



Power spectral density $S(f)$ calculated with parameters $\gamma = 0.0004$; $\sigma = 0.025$; $\epsilon = 0.07$; $\tau_m = 1$; $m = 6$. Straight lines approximate power spectrum $S \sim 1/f^{\beta_{1,2}}$ with $\beta_1 = 0.33$ and $\beta_2 = 0.72$

Poissonian-like point process

- We consider a model with the autoregressive change of the mean interevent time $\bar{\tau}$ superimposed by the Poisson distribution

$$P(\tau_k|\bar{\tau}) = \frac{1}{\bar{\tau}} \exp\left(-\frac{\tau_k}{\bar{\tau}}\right)$$

of the actural interevent time τ_k

- For the mean interevent time we take the exponential diffusion reversion equation

$$\mathrm{d}\bar{\tau} = \left[\gamma - \frac{m}{2}\sigma^2 \left(\frac{\bar{\tau}}{\tau_m}\right)^m\right] \bar{\tau}^{2\mu-2}\mathrm{d}t + \sigma\bar{\tau}^{\mu-1/2}\mathrm{d}W$$

- The associated Fokker Planck equation with zero flow gives stationary PDF

$$P_t(\bar{\tau}) \sim \bar{\tau}^{\alpha+1} \exp\left[-\left(\frac{\bar{\tau}}{\tau_m}\right)^m\right]$$

with $\alpha = 2(\gamma_\sigma - \mu)$

Poissonian-like point process

- The long time distribution of interevent time τ_k has the integral form

$$P_t(\tau_k) = C \int_0^\infty \exp\left[-\frac{\tau_k}{\tau}\right] \tau^\alpha \exp\left[-\left(\frac{\tau}{\tau_m}\right)^m\right] \mathrm{d}\tau$$

with C defined from the normalization.

- In the case $m = 1$, PDF has a simple form

$$P_t(\tau_k) = \frac{2}{\Gamma(2+\alpha)\tau_m} \left(\frac{\tau_k}{\tau_m}\right)^{\frac{1+\alpha}{2}} K_{1+\alpha}\left(2\sqrt{\frac{\tau_k}{\tau_m}}\right)$$

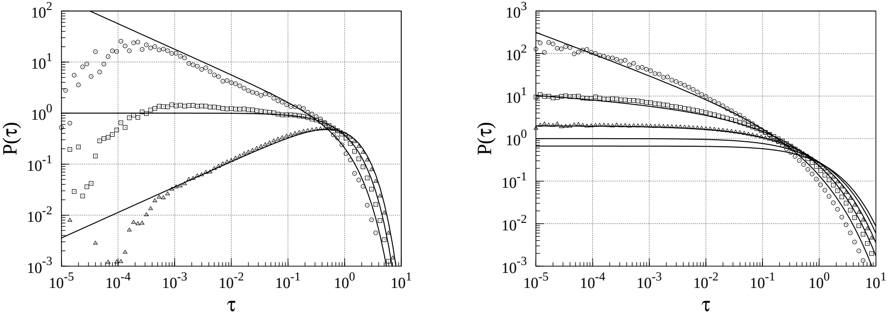
- In the k-space PDF of τ_k is

$$P_k(\tau_k) = \frac{2}{\Gamma(1+\alpha)\tau_m} \left(\frac{\tau_k}{\tau_m}\right)^{\frac{\alpha}{2}} K_\alpha\left(2\sqrt{\frac{\tau_k}{\tau_m}}\right)$$

Poissonian-like point process

- Limiting cases:
  1. $\tau_k \ll \tau_m$:
     1. $\alpha > 0$:
     $$P_k(\tau_k) = \frac{1}{\alpha \tau_m} = \mathrm{const}$$
     2. $\alpha = 0$:
     $$P_k(\tau_k) = \frac{1}{\tau_m}\left[\ln\frac{\tau_m}{\tau_k} - 2C\right], \qquad C = 0.577216\ldots$$
     3. $\alpha < 0$:
     $$P_k(\tau_k) = \frac{\Gamma(|\alpha|)}{\Gamma(1+\alpha)\tau_m}\left(\frac{\tau_k}{\tau_m}\right)^{\alpha}, \qquad -1 < \alpha < 0$$
  2. $\tau_k \gg \tau_m$:
  $$P_k(\tau_k) = \frac{2\sqrt{\pi}}{\Gamma(1+\alpha)\tau_m}\left(\frac{\tau_k}{\tau_m}\right)^{\frac{\alpha}{2}-\frac{1}{4}} \exp\left\{-2\sqrt{\frac{\tau_k}{\tau_m}}\right\}$$

Poisson-like point process



Summary

- The generalized multiplicative point processes generate time series exhibiting the power spectral density $S(f) \sim 1/f^{\beta}$ with $0.5 < \beta < 2$, i.e., with the slope observable in a large variety of systems.
- Such spectral density is caused by the stochastic diffusion of the average interpulse time, resulting in the power-law distribution.
- This yields the power-law distribution of the stochastic signal, $P(I) \sim 1/I^{\lambda}$ with $2 < \lambda < 4$, i.e., the phenomenon observable in a large variety of processes, from earthquakes to the financial time series.
- The set of the nonlinear stochastic differential equations for the signal intensity generating signals with $1/f^{\beta}$ noise and $1/x^{\lambda}$ distribution density of the intensity with different values of β and λ is derived.

Thank you for your attention!