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Continuous transition from the extensive to the non-extensive statistics in an agent-based herding model

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Abstract. Systems with long-range interactions often exhibit power-law distributions and can by described by the non-extensive statistical mechanics framework proposed by Tsallis. In this contribution we consider a simple model reproducing continuous transition from the extensive to the non-extensive statistics. The considered model is composed of agents interacting among themselves on a certain network topology. To generate the underlying network we propose a new network formation algorithm, in which the mean degree scales sub-linearly with a number of nodes in the network (the scaling depends on a single parameter). By changing this parameter we are able to continuously transition from short-range to long-range interactions in the agent-based model.

1 Introduction

Properties of systems with long-range interactions concern a wide range of problems in physics [1]: gravitational forces [2] and Coulomb forces in globally charged systems [3], vortices in two-dimensional fluid mechanics [4], wave-particles interaction [5], and trapped charged particles [6]. Such systems are of particular interest because they violate extensivity and additivity, two basic properties used to derive the thermodynamics of a system. Consequently they have been a subject of extensive studies in the recent years (for reviews see [7,8]). Small systems, in which the range of interactions is comparable to the size of the system, are also non-additive and thus are similar to large systems with truly long-range interactions. These systems can exhibit novel types of behavior – e.g., inequivalence of the microcanonical and canonical ensembles [9] and negative microcanonical specific heat [10]. Models with long-range interactions often possess dynamical features like slow relaxation [1,9] and broken ergodicity [9,11]. Another characteristic feature is the emergence of long-lived non-equilibrium quasistationary states (QSS) and violent relaxation into these states [12]. Non-Gaussian distributions [13] and non-exponential relaxations for autocorrelations [14] have been observed as well.

Non-extensive statistical mechanics is intended to describe some of the systems with long-range interactions by generalizing the Boltzmann-Gibbs statistics [15–17]. There are systems that, depending on the initial conditions, are not ergodic in the entire phase space and may prefer a particular subspace. If that subspace has a scale invariant geometry, a hierarchical or multifractal structure, then the model points toward non-extensive statistical mechanics. The generalized statistical mechanics framework is based on a generalized entropy [15], which is assumed to be given by

$$S_q = \left(1 - \int [p(x)]^q dx\right)/(q-1), \qquad (1)$$

where $p(x)$ is a probability density function of finding the system in the state characterized by the parameter x, while q is a parameter describing the non-extensiveness of the system. In the limit $q \to 1$ the traditional Boltzmann-Gibbs entropy is recovered from equation (1) [15,16]. Concepts drawn from this generalized framework have found their applications in a variety of traditional disciplines, such as physics [18–20], chemistry, biology or economics, and also in an interdisciplinary field of the complex systems [21–23].

Consequences of long-range interactions usually have been investigated in Hamiltonian systems. In this paper we explore long-range interactions in agent-based modeling instead. Agent-based modeling is one the most prominent contemporary tools used to obtain insights into the complex socio-economic systems. It is the main tool used to model opinion dynamics [24,25], explain emergent phenomena in microeconomics [26] and macroeconomics [27,28], reproduce the dynamics observed in the financial markets [29,30] and solve logistic problems for the business practitioners [31]. Some approaches starting from agent-based modeling obtain non-linear stochastic differential equations (SDEs) as a macroscopic model for the underlying agent-based dynamics [29,32–34], thus providing microscoping reasoning for the socio-economic dynamics. Another layer of understanding may be provided by another contemporary tool known as network theory,

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which allows to uncover the intrinsic relationships in geological [35], biological [36], socio-economic [37,38] and other complex systems [39,40].

In the context of this contribution the most interesting approaches are based on the agent-based herding model, originally proposed and developed in a series of papers by Kirman and Teyssière [41–43], as these approaches are able to reproduce both the power-law and Gaussian-like distributions [37,44]. In reference [37] it was shown that Kirman's model reproduces power-law distribution if the underlying model topology is a random network, but if the topology is a small-world or a scale-free network, then the Gaussian-like distribution is obtained. This result can be easily understood by looking into the scaling of each network's mean degree. The network where the mean degree $\langle d\rangle$ is fixed, $\langle d\rangle \sim$ const., (e.g., small-world or a scale-free network) represents short-range interactions, whereas the network where the mean degree scales linearly with the number of nodes, $\langle d\rangle \sim N$, (e.g., a random network) represents truly long-range interactions and corresponds to Hamiltonian mean-field models.

In this paper we connect those two extreme cases by proposing a new network formation model, which exhibits a sub-linear scaling of the mean degree, $\langle d\rangle \sim N^{\alpha}$ (with $\alpha \in [0, 1]$). By changing the single network parameter α we can continuously transition from short-range to long-range interactions in our agent-based model. This network formation model connects random and scale-free networks and can be useful in describing socio-economical systems.

The paper is organized as follows. In Section 2 we describe an extensive agent-based model corresponding to short-range interactions between the agents. To investigate the transition to long-range interactions we consider agent-based model implemented on a network. The network formation model is discussed in Section 3. This model is able to produce hybrid networks in between well-known random and scale-free networks and exhibits sub-linear scaling of the mean degree with the increasing number of nodes. In Section 4 we investigate an agent-based model implemented on this network. For this model the detailed network structure is not important and mean-field approximation yields a good result. Section 5 summarizes our findings.

2 Extensive agent-based model

We consider an agent model similar to the model proposed by Kirman (see [42]). There is a fixed number of agents, N, each of them being in state 1 or in state 2. In this model dynamic evolution is described as a Markov chain, the agents switch state either due to idiosyncratic factors or under the influence (e.g., peer pressure) of other agents. The lack of memory of the agents is the crucial assumption to formalize the dynamics as a Markov process. Describing the dynamics as a jump Markov process in a continuous time, we choose η_1 and η_2 to represent per-agent transition rates to the state written in the subscript. Namely, η_1 is a transition rate from state 2 to state 1. By choosing n to represent a whole number of agents in state 1, it becomes convenient to obtain a number of agents in state 2 via $N-n$. The aforementioned transition rates η_1 and η_2 can depend on n, $N-n$ as well as on the total number of agents N.

We can write the aggregate transition rates for one agent switching as

$$p(n \to n+1) \equiv p^{+}(n) = (N-n)\eta_1, \tag{2}$$

$$p(n \to n-1) \equiv p^{-}(n) = n\eta_2. \tag{3}$$

The above probabilities define a one-step stochastic process [45]. The transition probabilities imply the Master equation for the probability $P_n(t)$ to find n agents in the state 1 at time t [45]:

$$\frac{\partial}{\partial t}P_n = p^{+}(n-1)P_{n-1} + p^{-}(n+1)P_{n+1} - \left(p^{+}(n) + p^{-}(n)\right)P_n. \tag{4}$$

For large enough N we can represent the macroscopic system state by using a continuous variable $x = n/N$. Using the birth-death process formalism [45], one can obtain a non-linear Fokker-Planck equation from the Master equation (4) assuming that N is large and neglecting the terms of the Taylor expansion of the order of $1/N^2$:

$$\frac{\partial}{\partial t}P_x(x,t) = \frac{\partial}{\partial x}[x\eta_2 - (1-x)\eta_1]P_x(x,t) + \frac{1}{2N}\frac{\partial^2}{\partial x^2}[(1-x)\eta_1 + x\eta_2]P_x(x,t). \tag{5}$$

Taking into account the diffusion term, the steady state solution of the Fokker-Planck equation (5) is

$$P_0(x) = \frac{C}{(1-x)\eta_1 + x\eta_2} \times \exp\left[-2N\int^{x}\frac{x'\eta_2 - (1-x')\eta_1}{(1-x')\eta_1 + x'\eta_2}dx'\right]. \tag{6}$$

When the interactions between agents are short-range (in other words agents interact in their fixed size local neighborhood), the model is extensive and the transition rates η_1 and η_2 depend only on the continuous system state variable $x = n/N$ and do not directly depend on total number of particles N: $\eta_1 = \eta_1(x)$ and $\eta_2 = \eta_2(x)$. In the thermodynamic limit, when $N \to \infty$, we can neglect the diffusion therm in equation (5). In that case we get

$$\frac{\partial}{\partial t}P_x = \frac{\partial}{\partial x}[x\eta_2 - (1-x)\eta_1]P_x \tag{7}$$

with the corresponding steady state solution

$$P_0(x) = \delta(x - x_0). \tag{8}$$

Here x_0 is the solution of the equation describing the detailed balance:

$$x_0\eta_2(x_0) = (1-x_0)\eta_1(x_0). \tag{9}$$

Taking into account the diffusion term the steady state solution is given by equation (6). When $x' = x_0$ then the expression in the integral in equation (6) is zero. Expanding the expression in the integral around the point $x' = x_0$ and keeping only first-order term we get

$$P_0(x) \approx C' \exp\left[-2N\int^{x} A(x'-x_0)dx'\right] = \sqrt{\frac{NA}{\pi}}\exp\left[-NA(x-x_0)^2\right] \quad (10)$$

where the expansion coefficient A is

$$A = g'(x_0) + \frac{1}{2x_0(1-x_0)} \quad (11)$$

with

$$e^{2g(x)} \equiv \frac{\eta_2(x)}{\eta_1(x)}. \quad (12)$$

We obtain that the steady state probability distribution function (PDF) is approximately Gaussian with the width proportional to $1/\sqrt{N}$. This result is in agreement with the research presented by Traulsen et al. [46–48], who studied a very similar, yet significantly narrower (fixed form of η_i), case.

In order to investigate the effects of long-range interactions and non-extensivity in the agent model we need to have the transition rates η_1 and η_2 that explicitly depend on the total number of agents N. To construct the model that can have a practical relevance we will consider an agent-based model implemented on a network. We start by proposing a new network formation model in the following section.

3 Network formation model exhibiting sub-linear scaling of the mean degree

In this section we propose a new network formation model exhibiting sub-linear scaling of the mean degree $\langle d \rangle$ with the increasing number of nodes N in the network. To construct our network formation model we have chosen the Barabasi-Albert model [49] as our base model. We extend this model by adding an additional step. This means that during the first step in our network formation model we add a new node to the network and connect it to one old node based on the linear "rich gets richer" scheme. During the additional step the new node may form additional links with the immediate neighbors of the old node, the one it was connected to during the first step, with probability

$$p = p_0 d^{-\gamma}, \quad (13)$$

where p_0 is a probability to make a random connection when $\gamma = 0$, d is a degree of the old node, γ is a probability scaling exponent, which is related to the mean degree scaling exponent, α. An exemplary schema of the proposed formation model is shown in Figure 1.

Note that the additional step is somewhat similar to the techniques used in the triad formation [50,51], friends of friends [52] and forest fire [53] network formation models. As in the works [50–52] the additional links are formed only with the immediate neighbors of the old node. Though unlike in reference [52] we use Barabasi-Albert model as a base model. We also add a random amount of links during the additional step unlike the models considered in references [50–52]. The forest fire algorithm [53] also adds a random number of links, but it considers $\gamma = 0$ case. In the forest fire algorithm the mean degree scaling is achieved not by scaling the probability of forming the additional links, but by repeating the additional step until no new links are formed. Note that there are more network formation models, which exhibit sub-linear scaling of the mean degree, but mostly they are overly general and lack connections to the actual processes in the socio-economic systems [54–56].

Dependence of the mean degree $\langle d \rangle$ on the number of nodes in the network N for various values of the parameter γ is shown in Figure 2. Our numerical calculations indicate that

$$\alpha \approx (1-\gamma)^2 \quad (14)$$

for $\gamma \in [0, 1]$.

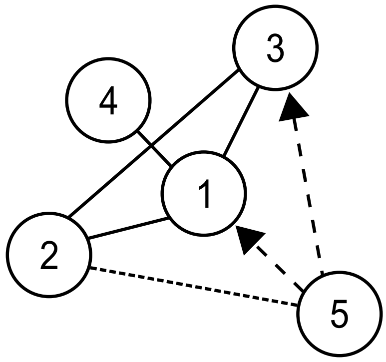

Fig. 1. Node 5 joins an existing network by making connection to Node 2 via the "rich gets richer" scheme (dashed line without arrows). After making this initial connection to the network, with a certain probability given by equation (13) Node 5 may connect (dashed arrows) to the neighbors of Node 2 (Node 1 and Node 3). Node 4 remains intact as it is not a direct neighbor of Node 2.

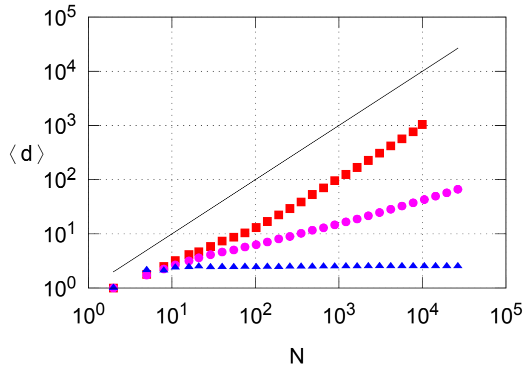

Fig. 2. Mean degree scaling for different values of γ: 0 (red squares), 0.3 (magenta circles), 1 (blue triangles). Black curve shows the mean degree scaling in completely connected network.

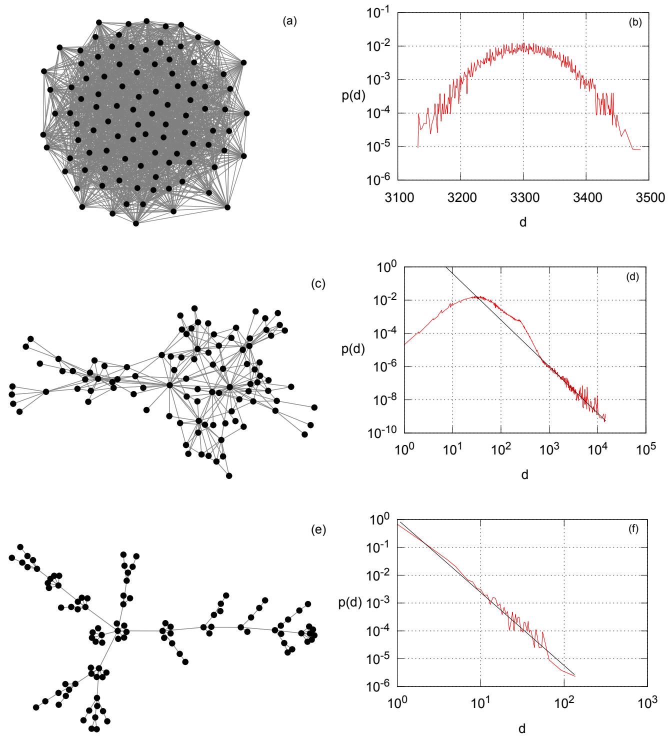

Fig. 3. Random network (topology (a), degree distribution (b)), scale-free network (topology (e), degree distribution (f)) and hybrid network (topology (c), degree distribution (d)) generated using the proposed network formation algorithm. Network snapshots (a), (c), (e) were taken at $N = 100$. Degree PDFs were obtained on networks with $N = 10^4$ (random network) and $N = 3 \times 10^4$ (hybrid and scale-free networks). Black curves in (d) and (f) provide power law fit (with exponent $\lambda = 3$) for the tail of the degree PDF. Following parameters were used: $p_0 = 0.3$, $\gamma = 0$ (random network), 0.3 (hybrid network), 1 (scale-free network).

In Figure 3 we demonstrate a transition between random and scale-free networks obtained using our network formation algorithm. With small values of γ we observe a randomly connected clump of nodes, we also observe Gaussian-like degree distribution in this clump. Slightly larger γ values allow for large degree hubs to form, while apparently random links are still present. While from $\gamma \gtrsim 1$ the probability of forming random links becomes small, thus random links disappear and only scale-free structure remains. Mean degree scaling for the same values of γ is shown in Figure 2.

4 Agent-based model executed on the network structure

In this section we consider Kirman's agent-based model implemented on a network generated using algorithm described in the previous section. There is a fixed number of agents, N, located on the nodes of the network, each of them being in state 1 or in state 2. Note, as the comparison of mean-field approximation with the exact solution shows, that the detailed network structure for this model is not important. The main influence of the network is via the scaling of the mean degree $\langle d \rangle$ with the number of nodes N in the network. Describing the dynamics as a jump Markov process in a continuous time, the transition probabilities per unit time for agent i being in the state X ($X = 1, 2$) to switch state to the other state Y ($Y \neq X$) are given by:

$$p_i(X \to Y) = \sigma + h n_i(Y), \tag{15}$$

where σ is the idiosyncratic switching rate, h describes the herding tendency and $n_i(Y)$ is the number of neighbors in the state Y.

4.1 Mean-field approximation

The mean-field approach for the model yields the following mean per-agent transition (from state X to state Y) rates [37]:

$$\langle p_i(X \to Y) \rangle = \sigma + h\langle d \rangle \frac{N_Y}{N}, \tag{16}$$

where N_Y is a total number of agents in the state Y. Using the notation introduced in Section 2 we would have $\eta_1 = \langle p_i(2 \to 1) \rangle$ and $\eta_2 = \langle p_i(1 \to 2) \rangle$.

Note that in the infinitely large system limit, $N \to \infty$, the herding behavior term disappears if $\langle d \rangle \sim$ const., while it remains constant if $\langle d \rangle \sim N$. If the herding term disappears, or becomes negligible, then the mean behavior of system becomes deterministic and only a small Gaussian-like fluctuations occur (see Sect. 2), while otherwise the power-law distribution is obtained [37,44].

The Fokker-Planck equation (5) for the model now becomes

$$\frac{\partial}{\partial t} P_x(x,t) = -\frac{\partial}{\partial x}\sigma(1-2x)P_x(x,t) + \frac{1}{2N}\frac{\partial^2}{\partial x^2}(2h\langle d \rangle x(1-x) + \sigma)P_x(x,t). \tag{17}$$

The dynamics of the continuous macroscopic system state variable x can be modeled by the SDE corresponding to the Fokker-Planck equation (17):

$$dx = \sigma(1-2x)dt + \sqrt{\frac{1}{N}(2h\langle d \rangle x(1-x) + \sigma)}dW_t, \tag{18}$$

where W_t is a Wiener process. In reference [57] it has been shown that in the case when $\langle d \rangle \sim N$ the fluctuations of the ratio N_2/N_1, exhibit $1/f^\beta$ power spectral density in a wide region of frequencies growing with N. In particular, we have $1/f$ noise when $\sigma/(d_0 h) = 2$. This is not the case when $\alpha < 1$ because for $\alpha < 1$ in the limit of $N \to \infty$ the macroscopic fluctuations of x vanish.

4.2 Steady state distribution of agents

Now let us consider the steady state of this system of agents and investigate the probability density function

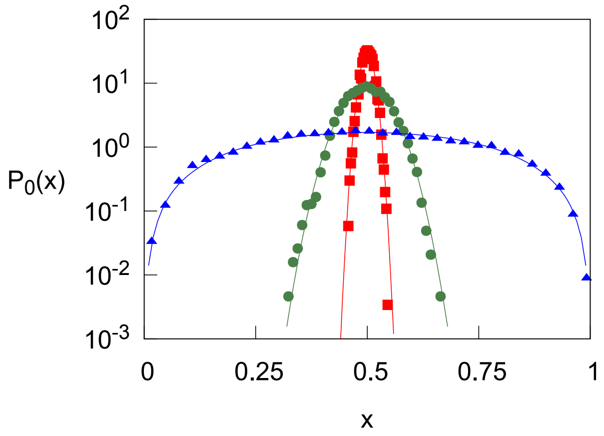

Fig. 4. Simulated steady state probability density function $P_0(x)$ for agent-based model with different values of mean degree exponent α: red squares $\alpha = 0$, green circles $\alpha = 0.5$, blue triangles $\alpha = 1$. Solid lines show the mean-field approximation of the steady state probability density function provided by equation (19). The parameter values of the model were $\sigma = 1.5$, $h = 1$, $N = 3000$, $p_0 = 0.75$, $\Delta t = 2 \times 10^{-5}$. The parameter γ values were $\gamma = 1$ ($\alpha = 0$), $\gamma = 0.3$ ($\alpha = 0.5$), $\gamma = 0$ ($\alpha = 1$). From the scaling of the mean degree $\langle d \rangle$ with changing N the following d_0 values were obtained: $d_0 = 3.2$ ($\alpha = 0$), $d_0 = 1.24$ ($\alpha = 0.5$) and $d_0 = 0.6$ ($\alpha = 1$).

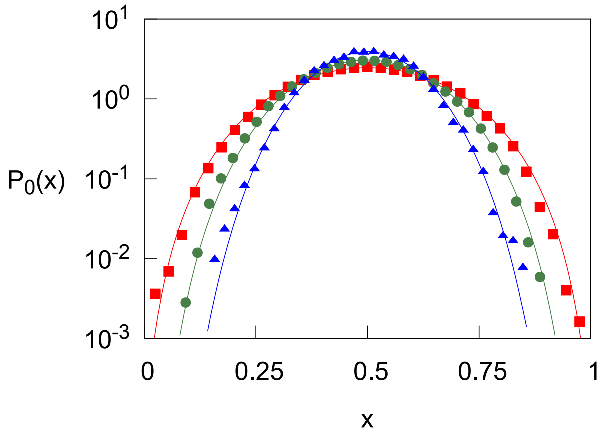

Fig. 5. Scaling of the simulated steady state probability density function $P_0(x)$ with the increasing number of agents in the model: red squares $N = 100$, green circles $N = 500$, blue triangles $N = 3000$. Solid lines show mean-field approximation of the steady state probability density function provided by equation (19). The remaining parameters of the model were $\sigma = 1.5$, $h = 1$, $p_0 = 0.75$, $\gamma = 0.15$, $\Delta t = 2 \times 10^{-5}$. The value of $d_0 = 0.9$ was obtained from the scaling of the mean degree $\langle d \rangle$ with changing N.

$P_0(x)$. If the mean degree $\langle d \rangle$ scales as N^α, that is $\langle d \rangle = d_0 N^\alpha$, the steady state PDF obtained from equation (17) according to equation (6) is

$$P_0(x) = C[\varepsilon + 2N^\alpha x(1-x)]^{\varepsilon N^{1-\alpha}-1}, \qquad (19)$$

where

$$\varepsilon \equiv \frac{\sigma}{d_0 h} \qquad (20)$$

and C is the normalization constant. The steady state PDF obtained from numerical simulation of the agent-based model described by equation (15) and comparison with the mean-field approximation (19) is shown in Figures 4 and 5. In the numerical simulations we choose a fixed time step Δt and consider transition probabilities equal to $p_i(X \to Y)\Delta t$. The time step must be chosen such that all transition probabilities should be between 0 and 1. For a given network structure, we synchronously update the state of each agent according to the transition probabilities. In the mean-field steady state PDF we use the parameter d_0 extracted from the scaling of the mean degree $\langle d \rangle$ of the network with the number of nodes N. We see a good agreement of the simulated PDF with the mean-field approximation. The width of the steady state PDF increases with increase of α, as is shown in Figure 4 and decreases with increase of the number of agents N, as is evident from Figure 5. In the limit of $N \to \infty$ the PDF $P_0(x)$ becomes very narrow if $\alpha < 1$.

Equation (19) can be rewritten in a q-Gaussian form

$$P_0(x) = C' \exp_q\left[-A_q\left(x - \frac{1}{2}\right)^2\right] \qquad (21)$$

with

$$q = 1 - \frac{1}{\varepsilon N^{1-\alpha} - 1}, \qquad A_q = 2N^{1-\alpha}\frac{1 - \frac{1}{\varepsilon}N^{\alpha-1}}{\frac{1}{2\varepsilon} + N^{-\alpha}}. \qquad (22)$$

The q-Gaussian PDF can be obtained by applying the standard variational principle on the generalized entropy (1) (see [15]). In the above $\exp_q(\cdot)$ is the q-exponential function, defined as:

$$\exp_q(x) \equiv [1 + (1-q)x]_+^{\frac{1}{1-q}}, \qquad (23)$$

here $[x]_+ = x$ if $x > 0$, and $[x]_+ = 0$ otherwise. The q-Gaussian steady state solution of the Fokker-Planck equation (17) can be explained by noting that equation (17) satisfies the condition given by equation (11) of reference [58]. The steady state PDF (21) having q-Gaussian form for finite values of N is in agreement with known results that Tsallis generalized canonical distribution describes systems in contact with a finite heath bath [59,60]. Equation (21) also confirms the similarity of small systems to large systems with truly long-range interactions.

If the interactions are long-range, $\alpha = 1$ and $\langle d \rangle \sim N$, and the system is infinitely large, $N \to \infty$, then the steady-state PDF (19) has a power-law form

$$P_0(x) = \frac{\Gamma(2\varepsilon)}{\Gamma(\varepsilon)^2}[x(1-x)]^{\varepsilon-1}. \qquad (24)$$

This corresponds to non-extensivity parameter

$$q = 1 - \frac{1}{\varepsilon - 1}. \qquad (25)$$

On the other hand, if interactions are short-range, $\alpha = 0$ and $\langle d \rangle \sim$ const., and the system infinitely large, $N \to \infty$, then according to equation (9), the steady-state PDF is Dirac delta function centered on $x_0 = 1/2$. As real systems are never infinite, for large N the steady-state PDF has a Gaussian-like form. If $\alpha < 1$ and N is large then q tends to 1 and from the properties of the q-exponential function we get that the steady state PDF (21) is approximately Gaussian

$$P_0(x) \sim \exp\left[-N^{1-\alpha}A\left(x-\frac{1}{2}\right)^2\right] \tag{26}$$

with

$$A = \begin{cases} \frac{2}{\frac{1}{2\varepsilon}+1}, & \alpha = 0 \\ 4\varepsilon, & 0 < \alpha < 1. \end{cases} \tag{27}$$

In this equation the coefficient A for $\alpha = 0$ is the same as given by equation (11). Thus the steady-state PDF retains its form in the $N \to \infty$ limit only if $\alpha = 1$, while in all other cases the N-dependence problem, considered by Alfarano and Milakovic [37], is obtained: namely the shape and variance of the distribution is lost with the increasing size of the system. It should be noted, that when $0 < \alpha < 1$, the fluctuations in the system decay not as $1/\sqrt{N}$, as it is usual in the statistics of extensive systems, but slower as $1/\sqrt{N^{1-\alpha}}$. The fluctuations decay slower with increasing N when α is closer to 1. In the limiting non-extensive case of $\alpha = 1$ the fluctuations do not decay at all with increasing the system size and are always macroscopic.

5 Conclusions

In summary, we have demonstrated a simple agent-based model that by changing the single parameter α can continuously transition from extensive to non-extensive statistics. Transition from extensive to non-extensive statistics in the agent-based model with changing the parameter α and the number of agents N is shown in Figure 6. As we can see, the extensive region becomes wider as N increases. However, for $\alpha = 1$ the behavior is non-extensive for all values of N.

The steady state distribution of agents for a finite system size is described by q-Gaussian (21) with $q \leq 1$. For $\alpha < 1$ and increasingly large system size (e.g. $N \to \infty$) the steady state distribution of the model tends to a Gaussian form with the width depending on α: as α increases the width decreases more slowly with increasing N. This simple model allows us to deepen the understanding of the effects of long-range interactions and observe the emergence of non-extensivity.

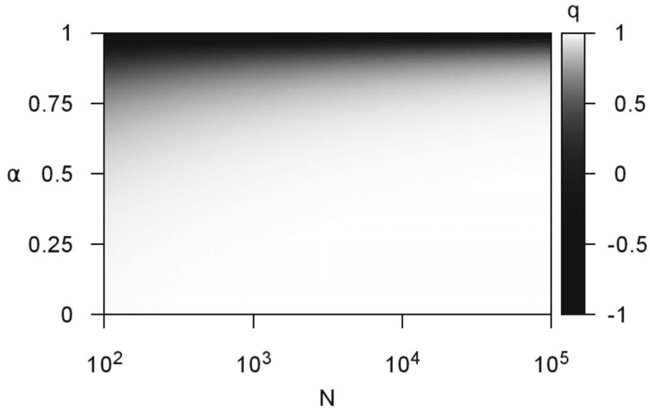

Fig. 6. Regions of extensive and non-extensive behavior of the agent-based model as described by the non-extensivity parameter q, equation (22). White color corresponds to extensive ($q \approx 1$), black color to non-extensive ($q < 1$) behavior. The parameter ε equals 1.5.

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