Intermittency generating $1/f$ noise

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***Abstract*—We analyze a mechanism of intermittecy in nonlinear dynamical systems having the invariant subspace and zero transverse Lyapunov exponent. Our model is similar to the on-off intermittency, occurring due the time-dependent forcing of a bifurcation parameter through a bifurcation point but with nonzero transverse Lyapunov exponent. We show that our nonlinear dynamical systems exhibit $1/f^\beta$ noise of the deviation from the invariant subspace. Further, the approximation of the intermittency generating maps by the nonlinear stochastic differential equations is presented and the connection with the equations modeling $1/f^\beta$ noise is established.**

I. Introduction

Intermittency is a random alternation of a signal between a quiescent state and bursts of activity, similar to the flicker process. Many natural systems display intermittent behavior, as, e.g., turbulent bursts in otherwise laminar fluid flows, sunspot activity, and reversals of the geomagnetic field. Well known models of intermittency include the three types introduced by Pomeau and Manneville [1], as well as crisis-induced intermittency [2]. A different variety of intermittency was first reported when synchronized chaos in a coupled chaotic oscillator system undergoes the instability when changing the coupling constant [3], [4]. This intermittency is known as the on-off intermittency [5]–[7]. The on-off intermittency appears in nonlinear dynamical systems with invariant subspaces, when the dynamics is restricted to the invariant subspace is chaotic and the system is close to a threshold of transverse stability of the subspace. The main difference of the on-off intermittency from other intermittency types is in the mechanism of the origin, i.e., the on-off intermittency relies on the time-dependent forcing of a bifurcation parameter through a bifurcation point, while in Pomeau-Manneville intermittency and crisis-induced intermittency the parameters are static.

Processes having the power spectral density (PSD) at low frequencies f of the form $S(f) \sim 1/f^\beta$ with β close to 1 are commonly referred to as $1/f$ noise, $1/f$ fluctuations, pink noise or flicker noise. Power-law distributions of spectra of signals with $0.5 < \beta < 1.5$ are ubiquitous in physics and in many other fields, including traffic in computer networks and financial markets [8]–[14]. Many models of $1/f$ noise are not universal because of the assumptions specific to the problem under consideration. Recently, the nonlinear stochastic differential equations (SDEs) generating signals with $1/f$ noise were obtained [15]–[18] starting from the point process model of $1/f$ noise [19]–[21].

Another model of $1/f$ noise involves a class of maps generating intermittent signals. It is possible to generate power-law distributions and $1/f$ noise from simple iterative maps by fine-tuning the parameters of the system at the edge of chaos [22], [23] where the sensitivity to initial conditions of the logistic map is a lot milder than in the chaotic regime, since the Lyapunov exponent is zero and the sensitivity to changes in initial conditions follows a power-law [24]. Manneville [25] showed that, tuned exactly, an iterative function can produce interesting behavior, power-laws and $1/f$ PSD. This mechanism for $1/f$ noise only works for type-II and type-III Pomeau-Manneville intermittency [26].

Here we consider a mechanism of intermittency, similar to the on-off intermittency, occurring in nonlinear dynamical systems with invariant subspace. In contrast to the on-off intermittency, we consider the case when the transverse Lyapunov exponent is zero. By relating nonlinear dynamics with the $1/f$ noise model based on the nonlinear SDEs we show that for such nonlinear dynamical systems the PSD of the deviation from the invariant subspace can have $1/f^\beta$ form in a wide range of frequencies.

II. Intermittency with zero transverse Lyapunov exponent

We consider the two-dimensional maps [5]:

$$x_{n+1} = F(x_n)\,, \qquad y_{n+1} = G(x_n, y_n)\,. \tag{1}$$

Here y is the deviation form the invariant subspace, while the function G has the property $G(x,0) = 0$ and, consequently, $y = 0$ is the invariant subspace. We assume that the dynamics $x_{n+1} = F(x_n)$ in (1) is restricted to the invariant subspace and it is chaotic. If the transverse Lyapunov exponent

$$\lambda_\perp = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \ln\left|\frac{\partial G(x_n,0)}{\partial y}\right| \tag{2}$$

along an orbit on the invariant subspace converges and is less than zero, then the invariant subspace is transversely stable with respect to this orbit.

We consider a case when $\partial G(x,0)/\partial y = 1$ and, consequently, the transverse Lyapunov exponent is zero [27]. Furthermore, we will assume that the two terms with the lowest powers in the expansion of the function $G(x,y)$ in the power series of y have the form

$$G(x,y) = y + g(x)y^\eta \tag{3}$$

with $\eta > 1$. This form satisfies the condition $\partial G(x,0)/\partial y = 1$.

We will consider the case where the function $g(x)$ in (3) is not constant and can acquire both positive and negative values. Thus the expansion (3) for small values of y_n leads to the the map

$$y_{n+1} = y_n + z_n y_n^\eta\,, \qquad \eta > 1\,, \tag{4}$$

with $z_n \equiv g(x_n)$. It should be noted that when $\eta = 1$, the map (4) becomes a multiplicative map $y_{n+1} = y_n(1 + z_n)$, which is essentially the same as the map considered in Ref. [6] for

ICNF2013

modeling of the on-off intermittency. The map (4) is similar to Pomeau-Manneville map

$$y_{n+1} = y_n + a y_n^{\eta} \pmod 1 \tag{5}$$

on the unit interval with one marginally unstable fixed point located at $y = 0$ [1]. The main difference from the map (4) is that in the Pomeau-Manneville map (5) the coefficient in the second term is static.

Let us consider the situation when $y_n > 0$. If $z_n < 0$ then the map (4) leads to the decrease of the deviation from the invariant subspace $y = 0$, whereas for $z_n > 0$ the deviation y grows. In contrast to systems with nonzero transverse Lyapunov exponent, the growth or decrease of the deviation is not exponential.

If the average of the variable z is positive, $\langle z\rangle > 0$, and there is a global mechanism of reinjection, the map (4) leads to the intermittent behavior. As in the on-off intermittency, the intermittent behavior appears due to the time-dependent forcing of a bifurcation parameter through a bifurcation point $z = 0$, thus the behavior described by map (4) can be considered as a kind of the on-off intermittency. However, the on-off intermittency is usually investigated in dynamical systems with nonzero transverse Lyapunov exponent.

For small durations of the laminar phase, one can approximate the map (4) replacing y_n in the second term on the right hand side by the initial value y_0. In such a case Eq. (4) describes a random walk with drift. Since the average displacement due to the diffusion grows as $\sqrt{t}$ and the displacement due to drift term is proportional to t, for small enough durations t the diffusion is more important than the drift. It is known that for the unbiased random walk the distribution of the first return times has the power-law exponent $-3/2$ [28]. Therefore, for small enough durations t one can expect to observe the power-law form, $t^{-3/2}$, of the probability density function (PDF) of the laminar phase durations, the same as in the on-off intermittency.

The first two terms in the expansion (3) do not allow to determine uniquely the PDF of the deviation y. In order to determine PDF of y and PSD of the series $\{y_n\}$, we need to take into account more terms in the expansion of the function $G(x, y)$ in the power series of y. One of the possibilities that we will consider is for the third term in the expansion to be equal to $\gamma y^{2\eta-1}$ (note, that $2\eta - 1 > \eta$ when $\eta > 1$), leading to the map

$$y_{n+1} = y_n + z_n y_n^{\eta} + \gamma y_n^{2\eta-1}\,. \tag{6}$$

A mechanism of reinjection operates at large values of y and does not change (6), written for small values of y close to the invariant subspace.

III. Numerical analysis

Let us consider a particular case of the map (1) with the function $G(x, y)$ whose behavior for small values of y is described by equation (6). We will investigate how the parameter γ influences PDF of y and PSD of the series $\{y_n\}$. Other parameters we keep fixed, choosing the value of the exponent $\eta = 2$, the average of the variable z_n, $\langle z\rangle = 5\times10^{-5}$, and the variance $\langle(z - \langle z\rangle)^2\rangle = 1$. The chosen value of the average $\langle z\rangle$ is close to the critical value for the onset of intermittency $\langle z\rangle = 0$ and is much smaller than the standard deviation of the variable z_n. As a mechanism of reinjection we use a reflection at $y = 0.5$, leading to the map

$$y_{n+1} = 0.5 - \left|y_n + z_n y_n^2 + \gamma y_n^3 - 0.5\right|. \tag{7}$$

As a map $x_{n+1} = F(x_n)$ in (1) we take the chaotic driving by a tent map

$$x_{n+1} = \begin{cases} 2x_n\,, & 0 \le x_n \le \frac{1}{2} \\ 2 - 2x_n\,, & \frac{1}{2} \le x_n \le 1. \end{cases} \tag{8}$$

The variable z_n with given average $\langle z\rangle$ and variance $\langle(z - \langle z\rangle)^2\rangle$ can be obtained from x_n using the equation

$$z_n = \sqrt{\frac{\langle(z-\langle z\rangle)^2\rangle}{\langle(x-\langle x\rangle)^2\rangle}}(x_n - \langle x\rangle) + \langle z\rangle\,. \tag{9}$$

For the tent map (8) the average and the variance are $\langle x\rangle = 0.5$ and $\langle(x - \langle x\rangle)^2\rangle = 1/12$, respectively.

The numerical results for maps described by Eqs. (7), (8), and (9) for different values of the parameter γ are shown in Fig. 1. We calculate the power spectral density directly, according to the definition, as the normalized squared modulus of the Fourier transform of the signal,

$$S(f) = \left\langle \frac{2}{N}\left|\sum_{n=1}^{N} y_n e^{-i2\pi f n}\right|^2 \right\rangle, \tag{10}$$

where the angle brackets $\langle\cdot\rangle$ denote averaging over realizations. We used the time series $\{y_n\}$ of the length $N = 10^9$ and averaged over 100 realizations with randomly chosen initial value y_0.

From Figs. 1(a), (d), and (g) we see that these maps indeed lead to intermittent behavior, where the laminar phases are changed by bursts of activity corresponding to the large deviations of the variable y from the average value. Also we can see that the duration of the laminar phase increases with decreasing γ. The PDF of the variable y, shown in Figs. 1(b), (e), and (h), has a power-law form for larger values of y, whereas for small values of y the PDF decreases exponentially. The exponent of the power-law part of the PDF increases with decreasing of the parameter γ. The PSD of the time series $\{y_n\}$, shown in Figs. 1(c), (f), and (i), has a power-law behavior for a wide range of frequencies. In particular, PSD has $1/f$ behavior in Fig. 1(f). The power-law interval in the PSD in Figs. 1(c), (f), and (i) is $10^{-8} \lesssim f \lesssim 10^{-4}$.

As the numerical examples show, map (6) for some values of the parameters yields time series with power-law PSD. The exponent of the power-law part of PSD is close to 1. In addition, the PDF of the deviation from the invariant subspace y for these values of parameters has a power-law part, in contrast to the on-off intermittency where the exponent in the PDF is close to -1 and $1/\sqrt{f}$ PSD.

IV. Approximation maps by the nonlinear stochastic differential equations

For obtaining analytical expressions of the PDF and PSD of the deviation y, we approximate the map (6) by a SDE. To obtain the SDE corresponding to the map (6) we replace the variable z_n by a random Gaussian variable, having the same

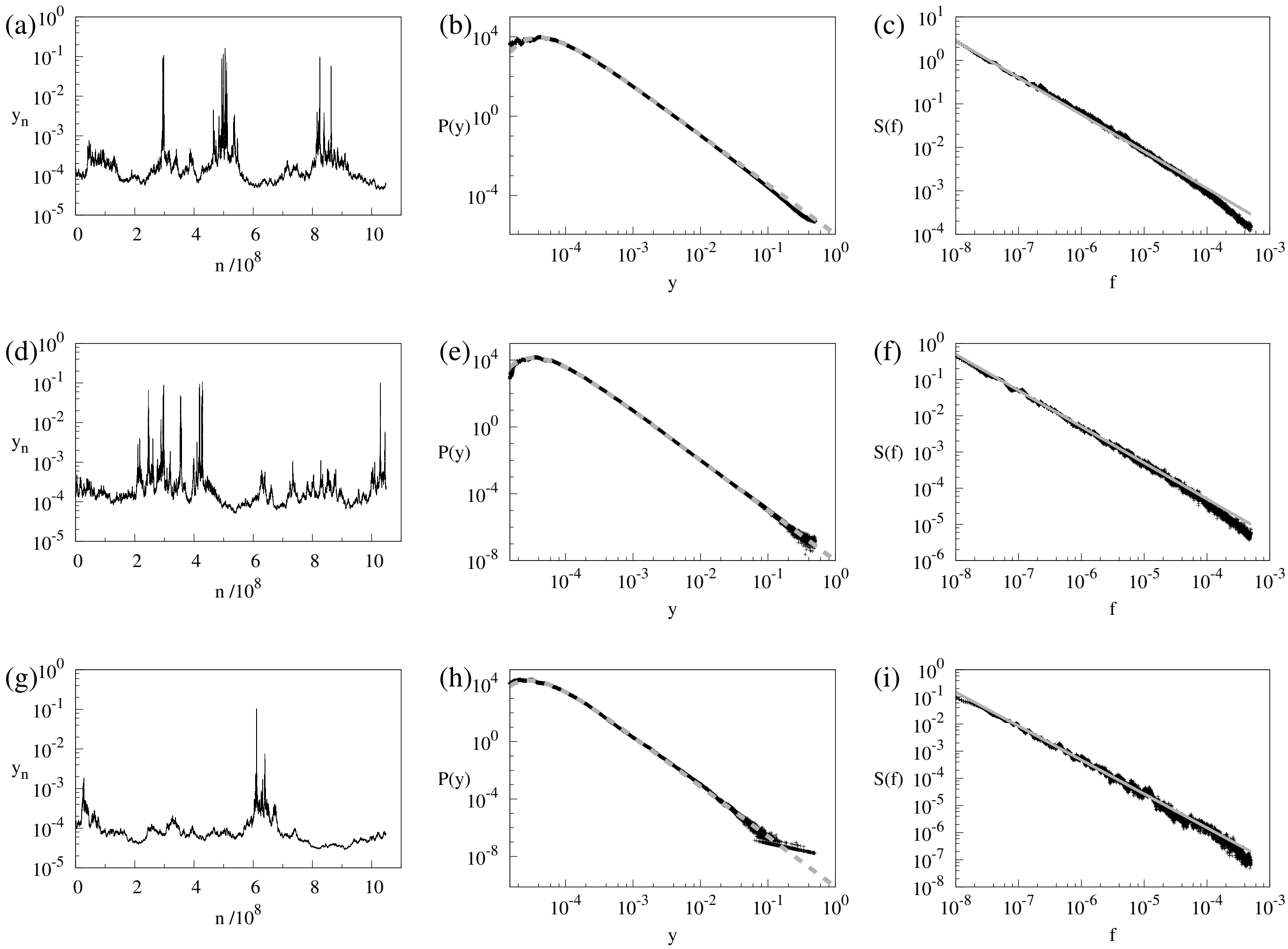

Fig. 1. Time series of map described by (7), (8), and (9) (a), (d), and (g), PDF of the variable y (b), (e), and (h), and PSD $S(f)$ of the time series of y (c), (f), and (i) for different values of the parameter γ. The parameter γ has the value $\gamma = 0.75$ in (a), (b),and (c), $\gamma = 0.5$ in (d), (e), and (f), and $\gamma = 0.25$ in (g), (h), and (i). The dashed line in (b), (e), and (h) is the analytical result (19) with the exponent $\nu = 2.5$ in (b), $\nu = 3$ in (e), and $\nu = 3.5$ in (h). The smooth gray line in (c), (f), and (i) shows the slope $1/f^{\beta}$ with $\beta = 0.85$ in (c), $\beta = 1$ in (f), and $\beta = 1.25$ in (i). Other parameters are $\langle z\rangle = 5 \times 10^{-5}$ and $\langle (z - \langle z\rangle)^2\rangle = 1$.

average and variance as z_n, and interpret Eq. (6) as Euler-Marujama approximation of a SDE. In this way we get the following SDE

$$dy = \sigma^2 \left(\eta - \frac{\nu}{2} + \frac{\eta - 1}{2} \left(\frac{y_{\min}}{y} \right)^{\eta - 1} \right) y^{2\eta - 1} dt + \sigma y^{\eta} dW\,. \tag{11}$$

Here W is a standard Wiener process (the Brownian motion) and the parameters σ, $y_{\min}$, and ν are given by the equations

$$\sigma = \sqrt{\langle (z - \langle z\rangle)^2\rangle}\,, \tag{12}$$

$$y_{\min} = \left[\frac{2\langle z\rangle}{(\eta - 1)\langle (z - \langle z\rangle)^2\rangle} \right]^{\frac{1}{\eta - 1}}\,, \tag{13}$$

$$\nu = 2\eta - \frac{2\gamma}{\langle (z - \langle z\rangle)^2\rangle}\,. \tag{14}$$

The SDE (11) has the same form as that considered in Refs. [15], [16]. The support for the nonlinear SDE of the form (11) starting from the agent-based herding model is presented in Ref. [29]. By introducing the additional parameters we can obtain the modified equations [30] modeling processes with q-exponential and q-Gaussian distributions of the nonextensive statistical mechanics.

The approximation of the map (6) by the SDE (11) is valid when the value of y is sufficiently small. The greatest value of y can be determined from the condition that the second term in Eq. (6) should be much smaller than the first one,

$$y_{\max}^{\eta} \sqrt{\langle (z - \langle z\rangle)^2\rangle} \ll y_{\max}\,. \tag{15}$$

giving

$$y_{\max} \lesssim \langle (z - \langle z\rangle)^2\rangle^{-\frac{1}{2(\eta - 1)}}\,. \tag{16}$$

Using Eqs. (13) and (16) we get the expression for the ratio $y_{\max}/y_{\min}$:

$$\frac{y_{\max}}{y_{\min}} \lesssim \left[\frac{(\eta - 1)\sqrt{\langle (z - \langle z\rangle)^2\rangle}}{2\langle z\rangle} \right]^{\frac{1}{\eta - 1}}\,. \tag{17}$$

As it was shown in Refs. [15], [16] SDE (11) generates signals with power-law PSD in a wide range of frequencies when the variable y can vary in a wide region, $y_{\max} \gg y_{\min}$. The condition $y_{\max}/y_{\min} \gg 1$ is obeyed when

$$\langle (z - \langle z \rangle)^2 \rangle \gg \langle z \rangle^2 \,, \tag{18}$$

that is, the standard deviation of the variable z_n should be much larger than the average.

SDE (11) leads to the steady state PDF

$$P_0(y) = \frac{(\eta - 1) y_{\min}^{\nu-1}}{\Gamma\left(\frac{\nu-1}{\eta-1}\right) y^{\nu}} \exp\left[-\left(\frac{y_{\min}}{y}\right)^{\eta-1}\right] . \tag{19}$$

Thus, the parameter ν gives the exponent of the power-law part of the PDF and the parameter $y_{\min}$ gives the position of the exponential cut-off at small values of y. From Eq. (13) it follows that $y_{\min}$ grows with the growing average $\langle z \rangle$. As can be seen in Figs. 1(b), (e), and (h), there is a good agreement of the numerically obtained PDF with the analytical expression (19). Similarly as in the case of the on-off intermittency we obtain PDF of the deviation from the invariant subspace having power-law form, but the power exponent ν can assume values significantly large then 1.

Numerical analysis indicates that the stochastic variable y, described by a SDE similar to (11) exhibits intermittent behavior, i.e., there are peaks or extreme events, corresponding to the large deviations of the variable from the average value, separated by laminar phases with a wide range distribution of the laminar durations. The exponent $-3/2$ in the PDF of the interburst durations has been obtained numerically [17] and analytically [31], as well.

In Refs. [15], [16] it was shown that SDE (11) generates signals with PSD having the form $S(f) \sim f^{-\beta}$ in a wide range of frequencies with the exponent

$$\beta = 1 + \frac{\nu - 3}{2(\eta - 1)} \, . \tag{20}$$

The connection of the PSD of the signal generated by SDE (11) with the behavior of the eigenvalues of the corresponding Fokker-Planck equation was analyzed in Ref. [18]. The power-law part of the numerically obtained PSD shown in Figs. 1 (c), (f), and (i) qualitatively agrees with (20). Eq. (20) gives correct exponent β for the parameters used in Figs. 1 (f), and (i), whereas for Fig. 1(c) it yields $\beta = 0.75$ instead of $\beta = 0.85$ obtained from numerical calculation. As long as the approximation of the map (6) by the SDE (11) is valid, the PSD of the time series $\{y_n\}$ exhibits a power-law behavior, including $1/f$ noise.

The range of frequencies where PSD has power-law behavior is limited by the minimum and maximum values $y_{\min}$ and $y_{\max}$. The limiting frequencies have been estimated in Ref. [18]. Using (12), (13) and (16), we can write the range of frequencies where PSD has the power-law form as

$$\left(\frac{y_{\min}}{y_{\max}}\right)^{2(\eta-1)} \ll 2\pi f \ll 1 \, . \tag{21}$$

Equation (21) demonstrates that the frequency region where PSD has $1/f^{\beta}$ dependence increases with raising of the ratio between minimum and maximum values of y and with increase of the difference $\eta - 1$ [18]. If $y_{\max}/y_{\min} \gg 1$, this frequency range can span many orders of magnitude. However, the estimation (21) of the frequency range is too broad and numerical solution of (11) gives narrower interval.

V. Conclusions

We revealed that the nonlinear maps having invariant subspace and the expansion in the powers of the deviation from the invariant subspace can generate signals with $1/f^{\beta}$ noise in wide range of frequencies. In contrast to Pomeau-Manneville type mechanism of $1/f$ noise, the exponent β of the PSD in our case depends on two parameters, thus $1/f^{\beta}$ noise can be obtained with various values of the exponent β. The width of the frequency region of such effect depends the average value of the variable z_n. The width increases when $\langle z \rangle$ approaches the threshold value $\langle z \rangle = 0$.

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