Modeling Gaussian and non-Gaussian 1/f noise by the linear stochastic differential equations

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***Abstract*—The ubiquitously observable $1/f$ noise is mostly Gaussian but sometimes the non-Gaussianity is recognizable, as well. Here we consider stochastic models of $1/f$ noise based on the linear stochastic differential equations with the very slowly varying coefficients (intensity of the white noise and relaxation rate) or consisting of a superposition of uncorrelated components with different distributions of these coefficients. We explore the conditions in which the modeled signal exhibiting $1/f^\beta$ noise is Gaussian and when it is non-Gaussian, i.e., the power-law distributed.**

I. Introduction

Abundance of $1/f$ distribution of the power spectral density have been found to be a key feature of a number of physical, natural, social, and biological phenomena [1], [2]. Mostly $1/f$ noise is considered as Gaussian process [2], [3], but in some cases the signals exhibiting $1/f$ fluctuations are non-Gaussian [4]–[9].

We have proposed stochastic models of $1/f^\beta$noise based on the nonlinear stochastic differential equations [10]–[14], starting from the point process model of $1/f$ noise [15], [16]. The models generate signals with the power-law distributions of signal intensity and power-law spectral density. There exist other models of $1/f$ noise, including the renewal point process with the power-law distribution of the interevent time [17] and random process represented by random pulses with randomly varying durations and amplitudes [18]–[21].

Nonetheless, $1/f$ noise is often modeled as a sum of independent Lorentzian spectra, resulting from uncorrelated signals with wide-range distributions of the relaxation times τ_{rel} [16], [22]. Distribution densities $P_l(x_l)$ of the signal components x_l modeled by the linear stochastic differential equations

$$dx_l = -\gamma_l x dt + \sigma_l dW_t \tag{1}$$

are Gaussian,

$$P_l(x_l) = \frac{1}{\sigma_l}\sqrt{\frac{\gamma_l}{\pi}}\exp\left[-\frac{\gamma_l}{\sigma_l^2}x_l^2\right]. \tag{2}$$

Here $\gamma_l = 1/\tau_l^{\mathrm{rel}}$ is the linear relaxation rate, σ_l is the white noise intensity, and W_t is the standard Wiener process (Brownian motion). The signal generated by Eq. (1) has the correlation function

$$C_l(s) = \frac{\sigma_l^2}{2\gamma_l}e^{-\gamma_l s}, \qquad s \geq 0 \tag{3}$$

and the power spectral density

$$S_l(f) = \frac{2\sigma_l^2}{\gamma_l^2+\omega^2}, \qquad \omega = 2\pi f. \tag{4}$$

The shapes of the probability density function $P(x)$ and of the power spectral density $S(f)$ of the signal represented as a sum of uncorrelated components $x_l(t)$,

$$x(t) = \sum_l x_l(t), \tag{5}$$

depend on the distribution of γ_l and σ_l [16], [23]. For instance, the distribution of relaxation rates $g(\gamma) \sim \gamma^{-1}$ together with the dependence $\sigma(\gamma) \sim \gamma^{1/2}$ yields $1/f$ noise of the Gaussian distributed signal (5) [16], while the uniform distribution of γ, $g(\gamma) = \mathrm{const}$, with $\sigma = \mathrm{const}$, results in $1/f$ noise with the power-law distribution of x, $P(x) \sim 1/x^3$ [23].

II. Modeling $1/f^\beta$ noise

In this paper we consider and analyze distributions and power spectra of signals (5) generated by the linear stochastic differential equations (1) or by equation

$$dx = -\gamma(t)xdt + \sigma(t)dW_t \tag{6}$$

with the very slowly fluctuating relaxation rate $\gamma(t)$ and intensity of the white noise $\sigma(t)$. This model is a generalization of the model based on the linear stochastic differential equation with the diffusion-like fluctuations of relaxation rate and constant intensity of the white noise [23],

$$dx = -\gamma(t)xdt + \sigma dW_t. \tag{7}$$

Many non-equilibrium systems exhibit spatial or temporal fluctuations of some parameter. Usually there are two time scales. One time scale corresponds to the relaxation dynamics of the system to a stationary state, while another time scale is related to the fluctuating evolution of parameters. A particular case is when the relaxation time needed for the system to reach stationarity is much smaller than the time scale at which the fluctuating parameter changes. In such a case, in the long-term the non-equilibrium system is described by a superposition of different local dynamics at different time intervals. This framework has been called superstatistics [24]–[28]. The superstatistical approach has been successfully applied for a varity of problems, like interactions between hadrons from cosmic rays [29], fluid turbulence [30]–[33], granular material [34], electronics [35], and economics [36]–[41]. Recently the superstatistical approach in combination with the stochastic model of $1/f$ noise has been used for modeling Tsallis distributions with the long-range correlations [14] and for simulation of the financial observables [42].

Our model presented in this paper may be considered as a specific case of the superstatistical approach, when the observable signal $x(t)$ in (5) is a superposition of the constituents

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generated by subsystems (1) with different parameters or the signal produced by a system with fluctuating parameters (6). In the superstatistical approach the distribution $P(x)$ of the signal $x(t)$ is a superposition of the conditional distribution $P(x|\gamma)$ and of the local distribution of the relaxation rate γ in the wide interval $\gamma \in [\gamma_1, \gamma_2]$ with $\gamma_1 \ll \gamma_2$, i.e., the density of the relaxation rates, $g(\gamma)$,

$$P(x) = \int_{\gamma_1}^{\gamma_2} P(x|\gamma) g(\gamma) d\gamma. \tag{8}$$

The power spectral density of signal $x(t)$ can be calculated in a similar manner [23]

$$S(f) = \int_{\gamma_1}^{\gamma_2} S(f|\gamma) g(\gamma) d\gamma. \tag{9}$$

Here $S(f|\gamma)$ is the spectral density of signal $x(t)$ for the fixed relaxation rate value γ.

We go beyond the fluctuation-dissipation theorem and the equipartition of energy theorem and introduce the relationship between relaxation rates and the intensity of the white noise in the most simple power-law way,

$$\sigma(\gamma) = \sigma_0 \gamma^{\mu}. \tag{10}$$

Here μ is a parameter that depends on the system.

There are some cases where the fluctuation-dissipation theorem cannot be applied. The violation of the fluctuation-dissipation theorem has been found in the finite dimensional spin glasses [43] and in the systems out of equilibrium [44]. The theoretical study of motion of colloidal particles being confined in a harmonic well and dragged by a shear flow also shows violation of the fluctuation-dissipation theorem [45].

We analyze Eqs. (1) and (6) as Ito stochastic differential equations

$$dx = a(x)dt + b(x)dW_t\,. \tag{11}$$

For calculation of the steady state probability density function of the signal x we use the Fokker-Planck equation. The Fokker-Planck equation for the probability density function $P(x)$ corresponding to the Ito equation (11) is [46]

$$\frac{\partial}{\partial t}P = -\frac{\partial}{\partial x}a(x)P + \frac{1}{2}\frac{\partial^2}{\partial x^2}b^2(x)P. \tag{12}$$

The steady state solution of Eq. (12) has the form

$$P(x|a,b) = \frac{C}{b^2(x)} \exp\left(\int_0^x \frac{2a(x_1)}{b^2(x_1)}dx_1\right). \tag{13}$$

For the fixed values of relaxation rate γ and white noise intensity $\sigma(\gamma)$ the steady state distribution of the signal intensity is simply a Gaussian distribution,

$$P(x|\gamma,\sigma) = \frac{1}{\sigma(\gamma)}\sqrt{\frac{\gamma}{\pi}} \exp\left[-\frac{\gamma}{\sigma^2(\gamma)}x^2\right], \tag{14}$$

while the power spectral density of the signal $x(t)$ intensity is simply the Lorentzian distribution,

$$S(f|\gamma) = \frac{2\sigma^2(\gamma)}{\gamma^2 + \omega^2}. \tag{15}$$

Recently we proposed a description of diffusion-like fluctuations of relaxation rate by the stochastic differential equation [23]

$$d\gamma = \sigma_\gamma \gamma^{-\frac{\eta}{2}} dW_t. \tag{16}$$

Here σ_γ determines the rapidity of change of the relaxation rate driven by the white noise with the factor $\gamma^{-\frac{\eta}{2}}$, which imposes the power-law steady state distribution of the relaxation rate γ, $g(\gamma) \sim \gamma^{\eta}$. Using Eq. (13) we get the stationary distribution of the relaxation rates

$$g(\gamma) = \begin{cases} \frac{(1+\eta)}{\gamma_2^{1+\eta} - \gamma_1^{1+\eta}}\gamma^{\eta}, & \eta \neq -1, \\ \left[\ln\left(\frac{\gamma_2}{\gamma_1}\right)\gamma\right]^{-1}, & \eta = -1. \end{cases} \tag{17}$$

From Eqs. (9), (10), (15), and (17) we obtain the power spectral density

$$S(f) = \frac{C_\eta}{\omega^{1-(\eta+2\mu)}}\Bigg\{\left[\frac{\gamma_2}{\omega}\right]^{1+\eta+2\mu}\Phi\left(-\frac{\gamma_2^2}{\omega^2}, 1, \frac{1+\eta+2\mu}{2}\right) - \left[\frac{\gamma_1}{\omega}\right]^{1+\eta+2\mu}\Phi\left(-\frac{\gamma_1^2}{\omega^2}, 1, \frac{1+\eta+2\mu}{2}\right)\Bigg\}. \tag{18}$$

Here $\Phi(z,s,a)$ is the Lerch's phi transcendent and C_η is the normalization constant

$$C_\eta = \frac{\sigma_0^2(1+\eta)}{(\gamma_2^{\eta+1} - \gamma_1^{\eta+1})}. \tag{19}$$

In the frequency range $\gamma_1 \ll f \ll \gamma_2$, the power spectral density (18) is well approximated by the expression

$$S(f) = \frac{\sigma_0^2(1+\eta)(2\pi)^{\eta+2\mu}}{2(\gamma_2^{\eta+1} - \gamma_1^{\eta+1})\cos\left(\frac{\pi(\eta+2\mu)}{2}\right)}\frac{1}{f^{1-(\eta+2\mu)}}, \quad |\eta+2\mu| < 1. \tag{20}$$

Therefore, when the condition $|\eta+2\mu| < 1$ is satisfied, then $1/f^{\beta}$ noise,

$$S(f) \sim 1/f^{\beta}, \qquad \beta = 1-(\eta+2\mu), \tag{21}$$

may be observed and the exponent β varies between 0 and 2. For $|\eta+2\mu| > 1$ the $1/f^{\beta}$ noise is not detectable.

From Eqs. (8), (10), (14), and (17) we obtain the steady state probability density function

$$P(x) = \begin{cases} C\frac{1}{x^{\lambda}}\left[\Gamma\left(\frac{\lambda}{2}, \frac{\gamma^{1-2\mu}x^2}{\sigma_0^2}\right)\right]\Bigg|_{\gamma_2}^{\gamma_1}, & \mu \neq \frac{1}{2}, \\ \frac{1}{\sigma_0\sqrt{\pi}}e^{-\frac{x^2}{\sigma_0^2}}, & \mu = \frac{1}{2}, \end{cases} \tag{22}$$

$$C = \frac{(1+\eta)}{\gamma_2^{\eta+1} - \gamma_1^{\eta+1}}\frac{\sigma_0^{\lambda-1}}{\sqrt{\pi}(1-2\mu)}. \tag{23}$$

Here C is the normalization constant and λ is the exponent of the power-law,

$$\lambda = 3 + \frac{2(\eta+2\mu)}{1-2\mu}. \tag{24}$$

If inequality $x_{\min} \ll x \ll x_{\max}$, where $x_{\min} = \sigma_0\gamma_2^{\mu-1/2}$ and $x_{\max} = \sigma_0\gamma_1^{\mu-1/2}$, is satisfied then for $\mu \neq 1/2$ the distribution $P(x)$ behaves like the power-law,

$$P(x) = \frac{(1+\eta)}{\sqrt{\pi}(\gamma_2^{\eta+1} - \gamma_1^{\eta+1})}\frac{\sigma_0^{\lambda-1}\Gamma(\frac{\lambda}{2})}{|1-2\mu|}\frac{1}{x^{\lambda}}. \tag{25}$$

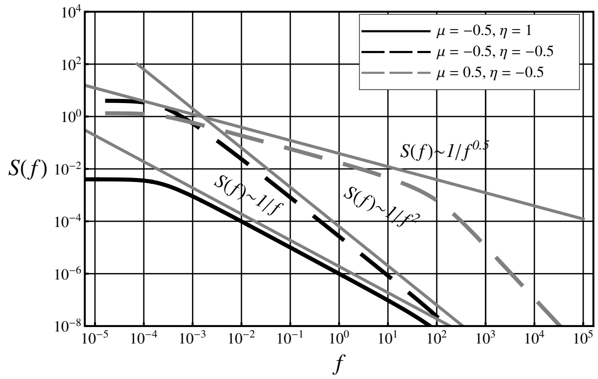

Fig. 1. Power spectral density $S(f)$ calculated using Eq. (18) with different values of parameters μ and η presented in the legend. Other parameters are: $\gamma_1 = 0.001$, $\gamma_2 = 1000$, and $\sigma_0 = 1$.

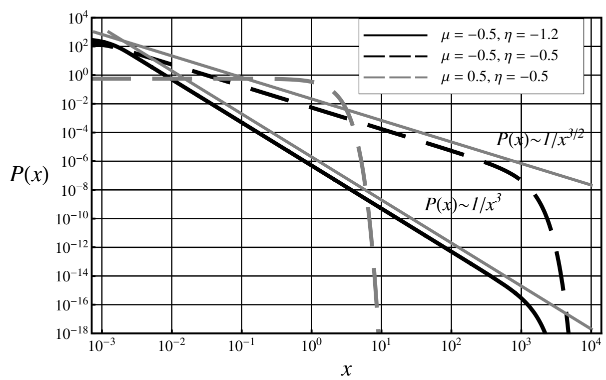

Fig. 2. Steady state probability density function $P(x)$ calculated using Eq. (22) with different parameter values μ and η presented in the legend. Other parameters are the same as in Fig. 1

Equation (25) approximates the probability density function (22) if conditions $\lambda > 0$ and $\mu \neq 0.5$ are satisfied.

The power spectral density and the probability density function of the signal $x(t)$ intensity with different values of the parameters η and μ are represented in Fig. 1 and Fig. 2. The black curves represent a case when η and μ satisfy conditions $|\eta + 2\mu| < 1$ and $\lambda > 0$. In this case we observe $1/f^{\beta}$ noise with $0 < \beta < 2$ and the power-law distribution of the signal, $P(x) \sim 1/x^{\lambda}$. The black dashed curves represent a case when $\lambda > 0$, but $|\eta+2\mu| > 1$. In such a case we detect the Brownian noise with $S(f) \sim 1/f^2$. However, the probability density is non-Gaussian and we observe the power-law unless $\mu = 1/2$. Just in a special case when $\mu = 1/2$ and $|\eta + 1| < 1$ we have the Gaussian $1/f^{\beta}$ noise (gray dashed curves).

III. Values of parameters μ and η for generation of $1/f^{\beta}$ noise with the power-law distribution of signal intensity

Our models (1) and (6) generate probability density function of signal intensity in the range between $x_{\min}$ and $x_{\max}$ as power-law, $P(x) \sim x^{-\lambda}$, for different parameters except $\mu = 0.5$. From Fig. 1 and Fig. 2 (black dashed curve) it is evident that for signals with the power-law probability density function, $1/f^{\beta}$ noise is not always observable. For some parameters the spectrum has $S(f) \sim 1/f^2$ tails (Fig. 1 black

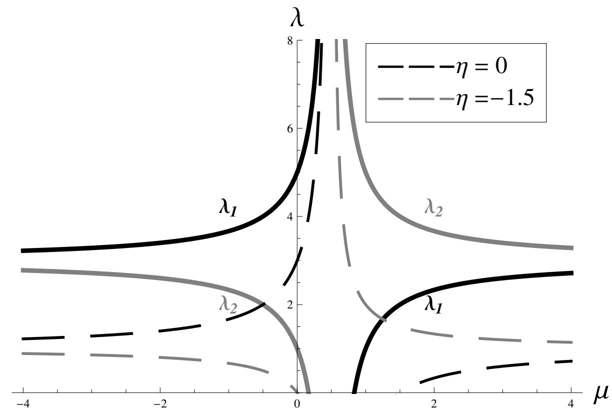

Fig. 3. The power-law exponent $\lambda(\mu)$, (24), dependence on the parameters μ and η. $\lambda_1(\mu)$ according to (27) (black curve) and $\lambda_2(\mu)$ according to (28) (gray curve) restrict an area $\lambda(\mu)$ where the condition $|\eta + 2\mu| < 1$ is satisfied. Only in the parts of $\lambda(\mu, \eta = \mathrm{const})$ between dashed curves $\lambda_1(\mu)$ and $\lambda_2(\mu)$ the condition $|\eta + 2\mu| < 1$ is satisfied and the power-law distribution is evident.

dashed curve). We now explore how to choose the values of the parameters η and μ in order to generate signals with the non-Gaussian $1/f^{\beta}$ noise and the possible values of the parameter λ. As it was shown above, $1/f^{\beta}$ noise with the power-law distribution of the signal may be generated when the condition $|\eta + 2\mu| < 1$ is satisfied, i.e., for $\eta > -1 - 2\mu$ or $\eta < 1 - 2\mu$. Thus we have $1/f^{\beta}$ noise with the power-law distribution if the parameter η belongs to the interval $\eta \in (\eta_1(\mu), \eta_2(\mu))$, where

$$\eta_1(\mu) = -1 - 2\mu, \qquad (26)$$
$$\eta_2(\mu) = 1 - 2\mu.$$

From relations (24) and (26) we obtain that $1/f^{\beta}$ noise with the power-law distribution of the signal is generated if the parameter λ belongs to the interval $\lambda \in (\lambda_1, \lambda_2)$, where

$$\lambda_1 = 3 - \frac{2}{1 - 2\mu}, \qquad (27)$$

$$\lambda_2 = 3 + \frac{2}{1 - 2\mu}. \qquad (28)$$

Functions $\lambda_1(\mu)$ and $\lambda_2(\mu)$ are plotted in Fig. 3. These functions restrict an area where the exponent $\lambda(\mu)$ satisfies the condition $|\eta + 2\mu| < 1$. Using curves $\lambda(\mu, \eta = \mathrm{const})$ we can plot the area between $\lambda_1(\mu)$ and $\lambda_2(\mu)$ where the exponent λ can have values between 0 and ∞ for the special choice of the parameters η and μ. Equations (21) and (24) yield the relationship

$$\beta = 1 + (\mu - 1/2)(\lambda - 3), \qquad (29)$$

similar to that obtained in the nonlinear models [12]–[14].

IV. Pure $1/f$ noise

From relation (20) it is obvious that the pure $1/f$ noise is generated when the condition $\eta + 2\mu = 0$ is satisfied, i.e., when

$$\eta = -2\mu. \qquad (30)$$

In the frequency range $\gamma_1 \ll f \ll \gamma_2$ Eqs. (22) and (30) yield signal with pure $1/f$ noise,

$$S(f) = \frac{\sigma_0^2(1+\eta)}{2(\gamma_2^{\eta+1} - \gamma_1^{\eta+1})}\frac{1}{f}, \qquad \eta = -2\mu, \tag{31}$$

and the probability density function

$$P(x) = \begin{cases} \frac{\sigma_0^2}{\sqrt{\pi}(\gamma_2^{1+\eta}-\gamma_1^{1+\eta})}\frac{1}{x^3}\left[\Gamma\left(\frac{3}{2}, \frac{\gamma^{1+\eta}x^2}{\sigma_0^2}\right)\right]\Bigg|_{\gamma_2}^{\gamma_1}, & \eta \neq -1, \\ \frac{1}{\sigma_0\sqrt{\pi}}e^{-\frac{x^2}{\sigma_0^2}}, & \eta = -1. \end{cases} \tag{32}$$

In the interval between $x_{\min} = \sigma_0\gamma_2^{-(1+\eta)/2}$ and $x_{\max} = \sigma_0\gamma_1^{-(1+\eta)/2}$ the probability density function (32) can be approximated as

$$P(x) = \frac{\sigma_0^2 \mathrm{sgn}(1+\eta)}{\pi(\gamma_2^{1+\eta} - \gamma_1^{1+\eta})}\frac{1}{x^3}. \tag{33}$$

V. Conclusion

We show that only in a special case, when $\langle x^2 \rangle = \sigma^2(\gamma)/(2\gamma) = \mathrm{const}$, i.e., when all fluctuators get equal average energy, we can obtain the Gaussian $1/f^\beta$ noise. In all other cases $1/f^\beta$ noise can be generated by the power-law distributed signal, $P(x) \sim 1/x^\lambda$, with $\lambda = 3$ corresponding to the pure $1/f$ noise, as in the nonlinear case [12]–[14]. Beyond the the equipartition theorem, the non-Gaussian $1/f^\beta$ noise may be observed.

The proposed model of $1/f^\beta$ noise extends our understanding of conditions for appearance of the phenomena in the complex systems with distinguishing long-range correlations and power-law distributions.

References

[1] A. Diniz, M. L. Wijnants, K. Torre, J. Barreiros, N. Crato, A. M. T. Bosman, F. Hasselman, R. F. Cox, G. C. V. Orden, and D. Delignières, *Human Movement Science*, vol. 30, p. 889, 2011.

[2] M. Li and W. Zhao, *Math. Probl. Egin.*, vol. 2012, p. 673648, 2012.

[3] S. Kogan, *Electronic Noise and Fluctuations in Solids*. Cambrige Univ. Press, 2008.

[4] V. Orlyanchik, M. B. Weissman, M. A. Torija, M. Sharma, and C. Leeghton, *Phys. Rev. B*, vol. 78, p. 094430, 2008.

[5] S. V. Melkonyan, *Physica B*, vol. 405, p. 379, 2010.

[6] I. Raicevic, D. Popovic, C. Panagopoulos, and T. Sasagawa, *Phys. Rev. B*, vol. 83, p. 195133, 2011.

[7] L. Zao and R. Coelho, *IET Signal Process.*, vol. 6, p. 684, 2012.

[8] S. Samanta and A. K. Raychaudhuri, *Phys. Rev. E*, vol. 85, p. 045127, 2012.

[9] P. V. Lin, X. Shi, J. Jaroszynski, and D. Popovic, *Phys. Rev. B*, vol. 86, p. 155135, 2012.

[10] B. Kaulakys and J. Ruseckas, *Phys. Rev. E*, vol. 70, p. 020101(R), 2004.

[11] B. Kaulakys, J. Ruseckas, V. Gontis, and M. Alaburda, *Physica A*, vol. 365, p. 217, 2006.

[12] B. Kaulakys and M. Alaburda, *J. Stat. Mech.*, vol. 2009, p. P02051, 2009.

[13] J. Ruseckas and B. Kaulakys, *Phys. Rev. E*, vol. 81, p. 031105, 2010.

[14] ——, *Phys. Rev. E*, vol. 84, p. 051125, 2011.

[15] B. Kaulakys, *Phys. Lett. A*, vol. 257, p. 37, 1999.

[16] B. Kaulakys, V. Gontis, and M. Alaburda, *Phys. Rev. E*, vol. 71, p. 051105, 2005.

[17] S. B. Lowen and M. C. Teich, *Fractal-Based Point Processes*. New Jersey: Wiley-Interscience, 2005.

[18] K. Schick and A. Verveen, *Nature*, vol. 251, p. 599, 1974.

[19] T. H. Bell, Jr., *J. Appl. Phys.*, vol. 45, p. 1902, 1974.

[20] J. Ruseckas, B. Kaulakys, and M. Alaburda, *Lithuanian. J. Phys.*, vol. 43, p. 223, 2003.

[21] ——, 2008, arXiv:0812.4674.

[22] J. Bernamont, *Ann. Phys. (Leipzig)*, vol. 7, p. 71, 1937.

[23] B. Kaulakys, M. Alaburda, and J. Ruseckas, *AIP Conf. Proc.*, vol. 922, p. 439, 2007.

[24] C. Beck and E. G. Cohen, *Physica A*, vol. 322, p. 267, 2003.

[25] C. Tsallis and A. M. C. Souza, *Phys. Rev. E*, vol. 67, p. 026106, 2003.

[26] S. Abe, C. Beck, and E. G. D. Cohen, *Phys. Rev. E*, vol. 76, p. 031102, 2007.

[27] M. G. Hahn, X. Jiang, and S. Umarov, *J. Phys. A: Math. Theor.*, vol. 43, p. 165208, 2010.

[28] C. Beck, *Philos. Trans. Royal Soc. A*, vol. 369, p. 453, 2011.

[29] G. Wilk and Z. Włodarczyk, *Phys. Rev. Lett.*, vol. 84, p. 2770, 2000.

[30] C. Beck, *Phys. Rev. Lett.*, vol. 87, p. 180601, 2001.

[31] C. Beck, E. G. D. Cohen, and H. L. Swinney, *Phys. Rev. E*, vol. 72, p. 056133, 2005.

[32] C. Beck, E. G. D. Cohen, and S. Rizzo, *Europhys. News*, vol. 36, no. 6, p. 189, 2005.

[33] C. Beck, *Phys. Rev. Lett.*, vol. 98, p. 064502, 2007.

[34] ——, *Physica A*, vol. 365, p. 96, 2006.

[35] F. Sattin and L. Salasnich, *Phys. Rev. E*, vol. 65, p. 035106(R), 2002.

[36] M. Ausloos and K. Ivanova, *Phys. Rev. E*, vol. 68, p. 046122, 2003.

[37] S. M. D. Queiros and C. Tsallis, *Europhys. Lett.*, vol. 69, p. 893, 2005.

[38] S. M. D. Queiros, *Physica A*, vol. 344, p. 619, 2004.

[39] S. M. D. Queiros and C. Tsallis, *Eur. Phys. J. B*, vol. 48, p. 139, 2005.

[40] S. M. D. Queiros, *Europhys. Lett.*, vol. 71, p. 339, 2005.

[41] J. de Souza, L. G. Moyano, and S. M. D. Queiros, *Eur. Phys. J. B*, vol. 50, p. 165, 2006.

[42] J. Ruseckas, V. Gontis, and B. Kaulakys, *Adv. Complex Syst.*, vol. 15, p. 1250073, 2012.

[43] E. Marinari, G. Parisi, F. Ricci-Tersenghi, and J. J. Ruiz-Lorenzo, *J. Phys. A: Math. Gen.*, vol. 31, p. 2611, 1998.

[44] D. Lobaskin and S. Kehrein, *J. Stat. Mech.*, vol. 123, p. 301, 2006.

[45] R. Mauri and D. Leporini, *Europhys. Lett.*, vol. 76, p. 1022, 2006.

[46] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences*. Berlin: Springer-Verlag, 2004.