Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

Special Issue on Unsolved Problems of Noise in Physics, Biology and Technology

Modeling of long-range memory processes with inverse cubic distributions by the nonlinear stochastic differential equations

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Received 19 October 2015
Accepted for publication 13 January 2016
Published 20 May 2016

Online at stacks.iop.org/JSTAT/2016/054035
doi:10.1088/1742-5468/2016/05/054035

Abstract. A well-known fact in the financial markets is the so-called 'inverse cubic law' of the cumulative distributions of the long-range memory fluctuations of market indicators such as a number of events of trades, trading volume and the logarithmic price change. We propose the nonlinear stochastic differential equation (SDE) giving both the power-law behavior of the power spectral density and the long-range dependent inverse cubic law of the cumulative distribution. This is achieved using the suggestion that when the market evolves from calm to violent behavior there is a decrease of the delay time of multiplicative feedback of the system in comparison to the driving noise correlation time. This results in a transition from the Itô to the Stratonovich sense of the SDE and yields a long-range memory process.

Keywords: models of financial markets, nonlinear dynamics, scaling in socio-economic systems, stochastic processes

1742-5468/16/054035+9$33.00

Contents

1. Introduction

Many systems show large fluctuations of macroscopic quantities that follow non-Gaussian, heavy-tailed, power-law distributions with power-law temporal correlations [1–3], scaling and fractal features [4–6]. The power-law distributions are often related both with the nonextensive statistical mechanics [7–9] and the power-law behavior of the power spectral density, i.e. $1/f^{\beta}$ noise 'ambiguity' (see, e.g. [4, 5, 9–12] and references therein).

One of the principal statistical features characterizing activity in the financial markets is the distribution of fluctuations of market indicators such as indexes. Frequently, long-range memory processes with heavy-tailed distributions and characteristic power-law exponents are observable. Power laws appear for relevant financial fluctuations, such as fluctuations of number of trades, trading volume and price. The well-known fact is the so-called 'inverse cubic law' of the cumulative distributions [2, 13–20], which is relevant to developed stock markets, to the commodity one, as well as to the most traded currency exchange rates. The exponents that characterize these power laws are similar for different types and sizes of markets, for different market trends and even for different countries [21–25]—suggesting that a generic theoretical basis may inspire these long-range memory inverse cubic phenomena. However, a general model of these phenomena remains an open issue.

One common way for describing all the above-mentioned forms of evolution is by means of the stochastic differential equations (SDEs) [10, 26–29]. These nondeterministic equations of motion are used in many systems of interest, such as simulating the Brownian motion in statistical mechanics, field theory models, financial systems, biology and many other areas.

Here, by the multiplicative SDEs [10, 29], we model the long memory processes with inverse cubic cumulative distributions. This is achieved taking into account a transition from the Itô to Stratonovich convention in noisy systems [30, 31], according to the Wong–Zakai theorem [32–35], with a decrease of the delay time of multiplicative feedback of the system, in comparison to the driving noise correlation time, when the market proceeds from calm to turbulent behavior.

2. Stochastic differential equations with variable convention parameter

We start from the simple quadratic SDE

$$\mathrm{d}x = x^2 \circ_\alpha \mathrm{d}W_t \tag{1}$$

where W_t is a Wiener process and α is the interpretation (convention) parameter, defining the α-dependent stochastic integral in equation (1),

$$\int_0^T f(x(t)) \circ_\alpha \mathrm{d}W_t \equiv \lim_{N\to\infty} \sum_{n=0}^{N-1} f(x(t_n))\Delta W_{t_n}. \tag{2}$$

Here $t_n = \frac{n+\alpha}{N}T$, $0 \leqslant \alpha \leqslant 1$. Common choices of the parameter α are: (i) $\alpha = 0$, pre-point (Itô convention), (ii) $\alpha = 1/2$, mid-point (Stratonovich convention) and (iii) $\alpha = 1$, post-point (Hänggi–Klimontovich, kinetic or isothermal convention) [31, 36, 37]. The quadratic SDE (1) is the simplest multiplicative SDE without a drift term and symmetric for the positive and negative deviations of the observable x.

Note that further in the paper when $\alpha = 0$, i.e. for the Itô convention, the symbol $\circ_\alpha$ is skipped in front of the differential $\mathrm{d}W_t$ of the Wiener process. All further SDEs, excluding (3) and (4), are the Itô equations. The Itô version of SDEs is the usual technique for modeling financial [38], economical, biological and many other systems [27]. This is reasonable since the Itô integral does not look into the future, it corresponds to the pre-action of the uncorrelated external noise on the nonlinear system. On the other hand, the Wong–Zakai theorem [32–34] states that the action of non-white noise with a finite correlation time can be approximated by the Stratonovich SDE version [30, 31, 35]. In real situations the external perturbation (noise) is always more or less correlated, particularly when the perturbations are strong. Note that the choice of SDE convention, Stratonovich or Itô, depends not only on the correlation time τ of the noise but also on the delay in the feedback δ [31].

The description of real phenomena by SDEs with multiplicative noise implies that the actual value of the observable influences the intensity of the noise action on the system and, therefore, the forthcoming state of the system. Similarly, the current value of the observable is influenced by its previous value. For example, the volatility of stock price may be the outcome of the influence of its past value [39–43]. More generally, such a process may be described by the delayed SDE [31]

$$\mathrm{d}x_t = G(x_t)\mathrm{d}t + F(x_{t-\delta})\zeta_t^\tau \mathrm{d}t. \tag{3}$$

Here $G(x_t)$ is a function representing the deterministic feedback of the system, yielding the drift of the observable x, $F(x_{t-\delta})$ is the multiplicative feedback (delayed by time δ) leading to the SDE with multiplicative noise, ζ_t^τ is a sufficiently regular correlated noise with the correlation time τ and δ is the delay time of multiplicative feedback of the system under consideration. It may be shown [31] that the limit, when $\delta \to 0$ and $\tau \to 0$ under the condition $\delta/\tau = \text{const}$, yields

$$\mathrm{d}x_t = G(x_t)\mathrm{d}t + F(x_t) \circ_\alpha \mathrm{d}W_t \tag{4}$$

with

$$\alpha\left(\frac{\delta}{\tau}\right) \simeq \frac{1}{2(1+\delta/\tau)}. \tag{5}$$

Consequently, in the case $\tau \ll \delta$ or for perturbation by white noise, $\tau \to 0$, even for short delay in feedback δ we achieve the Itô outcome, because there is no correlation between the sign of noise action ζ_t and the time derivative of the multiplicative feedback $F(x_t)$. In contrast, for very short delay and perturbation by the correlated noise, $\delta \ll \tau$, a correlation emerges between the sign of ζ_t and the time derivative of $F(x_t)$. This correlation yields the Stratonovich outcome [31–34].

3. Stochastic differential equations in financial systems

Equation (1) is a particular case of the more general equations. In order to obtain a stationary process and avoid divergence of the steady-state probability density function at $x = 0$, the diffusion of the stochastic variable x should be restricted, or equation (1) should be modified. Generalizations of this equation for different prefactors in front of the multiplicative noise and by introducing an additional parameter x_0 are presented in [9, 10]. The probability density function of the signal generated by modified Itô SDEs

$$\mathrm{d}x = \left(\eta - \frac{1}{2}\lambda\right)(x_0 + x)^{2\eta-1}\mathrm{d}t + (x_0 + x)^{\eta}\mathrm{d}W_t, \tag{6}$$

$$\mathrm{d}x = \left(\eta - \frac{1}{2}\lambda\right)(x_0^2 + x^2)^{\eta-1}x\mathrm{d}t + (x_0^2 + x^2)^{\eta/2}\mathrm{d}W_t \tag{7}$$

is the q-exponential and the q-Gaussian distribution of the nonextensive statistical mechanics [7–9], respectively. Here $\eta \neq 1$ is the exponent of the prefactor in front of the multiplicative noise and λ is the exponent of the asymptotic power-law, $P(x) \sim x^{-\lambda}$, steady-state probability density function of the stochastic variable x. SDEs (6) and (7) for small fluctuations, $|x| \ll |x_0|$, represent the linear additive stochastic process generating Brownian motion with steady drift or linear relaxation, respectively, and avoiding the divergence of the signal distribution when $x \to 0$, while for $|x| \gg |x_0|$ they reduce to the multiplicative SDE and preserve $1/f^{\beta}$ behavior of the power spectral density. In [9, 10] it was shown that SDEs (6) and (7) generate signals with power spectral density

$$S(f) \sim \frac{1}{f^{\beta}}, \qquad \beta = 1 + \frac{\lambda - 3}{2(\eta - 1)} \tag{8}$$

in a wide interval of frequencies. These equations have been used for modeling financial long memory observables with power-law and q-distributions [44]. Special options and generalizations of equations (6) and (7) take well-known forms in econophysics and finance [38, 45], modeling the familiar geometric Brownian motion (GBM), Bessel, squared Bessel, Cox–Ingersoll–Ross (CIR), constant elasticity of variance (CEV) processes and Marsh–Rosenfeld model [46, 47]. It should be noted that these models do not reproduce the long-range memory inverse cubic phenomena.

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4. Inverse cubic law for long-range correlated processes

Equation (1) is a particular case of SDEs (6) and (7) at $x_0 = 0$ and $\eta = 2$. More generally, the value of α in equation (1) may be variable and may depend on coordinate x and/or the system' parameters. Equation (1) with $\alpha \neq 0$ may be transformed to the Itô equation

$$\mathrm{d}x = 2\alpha x^3 \mathrm{d}t + x^2 \mathrm{d}W_t. \tag{9}$$

Equation (9) is a particular case with $\eta = 2$ of the general Itô equation [9, 10, 29],

$$\mathrm{d}x = \left(\eta - \frac{\lambda}{2}\right) x^{2\eta-1} \mathrm{d}t + x^{\eta} \mathrm{d}W_t, \;\; \eta \neq 1, \tag{10}$$

yielding the power-law steady-state, $P(x) \sim x^{-\lambda}$, distribution of signal with power-law spectrum (8). The relations between the parameters α and λ in equations (9) and (10) for $\eta = 2$ are:

$$\lambda = 4(1-\alpha), \qquad \alpha = 1 - \lambda/4. \tag{11}$$

Equation (10) and other similar random walk models are used for analysis of the Euro/Swiss franc exchange rate [48].

Figure 1 demonstrates statistics of solutions of equation (9) for different values of the parameter $\alpha = 0$; $1/4$; $1/2$ and 1, i.e. for $\lambda = 4$; 3; 2 and 0. We see that the inverse cubic cumulative distribution, $P_>(x) \sim x^{-\lambda+1}$, corresponding to $\lambda = 4$ yields the power spectral density $1/f^{\beta}$ distribution with $\beta \approx 1.5$, i.e. not a long-range correlated process.

It should be noted that for the cumulative inverse cubic distribution $P_>(x) \sim x^{-3}$, i.e. for $\lambda = 4$, according to equation (8) the exponent β is more than 1, $\beta > 1$, for all $\eta > 1$, therefore, the modeled processes (6), (7) and (10) are not long-range dependent. Note that the definition of the long-range correlated process corresponds to the power-law autocorrelation function $C(t) \sim 1/t^{\gamma}$ with $0 < \gamma < 1$, which takes place for $0 < \beta < 1$ and $\gamma = 1 - \beta$ [49].

Theoretically, in the range of validity of equations (8), $0.5 < \beta < 2$ [10, 29], the long-range correlated process with inverse cubic distribution may be obtained for $\eta < 0$, e.g. $\eta = -1/2$ yields $\beta = 2/3$. However, the corresponding SDE

$$\mathrm{d}x = -\frac{5}{2x^2}\mathrm{d}t + \frac{1}{\sqrt{x}}\mathrm{d}W_t, \tag{12}$$

as well as other equations with $\eta < 0$ can hardly be reasonable. The questions emerge: (i) why is the influence of the noise inversely proportional to the intensity of the process in some exponent, $x^{-|\eta|}$, and (ii) wherefore emerges the drift term $-(2-\eta)x^{-|2\eta-1|}$ with a very particular dependence on the intensity of the process $x^{-|2\eta-1|}$ and the specific coefficient $-(2-\eta)$? In contrast, the quadratic SDE (1) is without the poorly justified drift term and it is the simplest symmetric equation for the positive and negative deviations of the observable x.

For modeling the long-range dependent inverse cubic cumulative distribution with power spectral exponent $\beta < 1$ we take advantage of the characteristic features of the signal. Note that $1/f^{\beta}$ noise emerges due to large deviations of the signal, e.g. the flicker

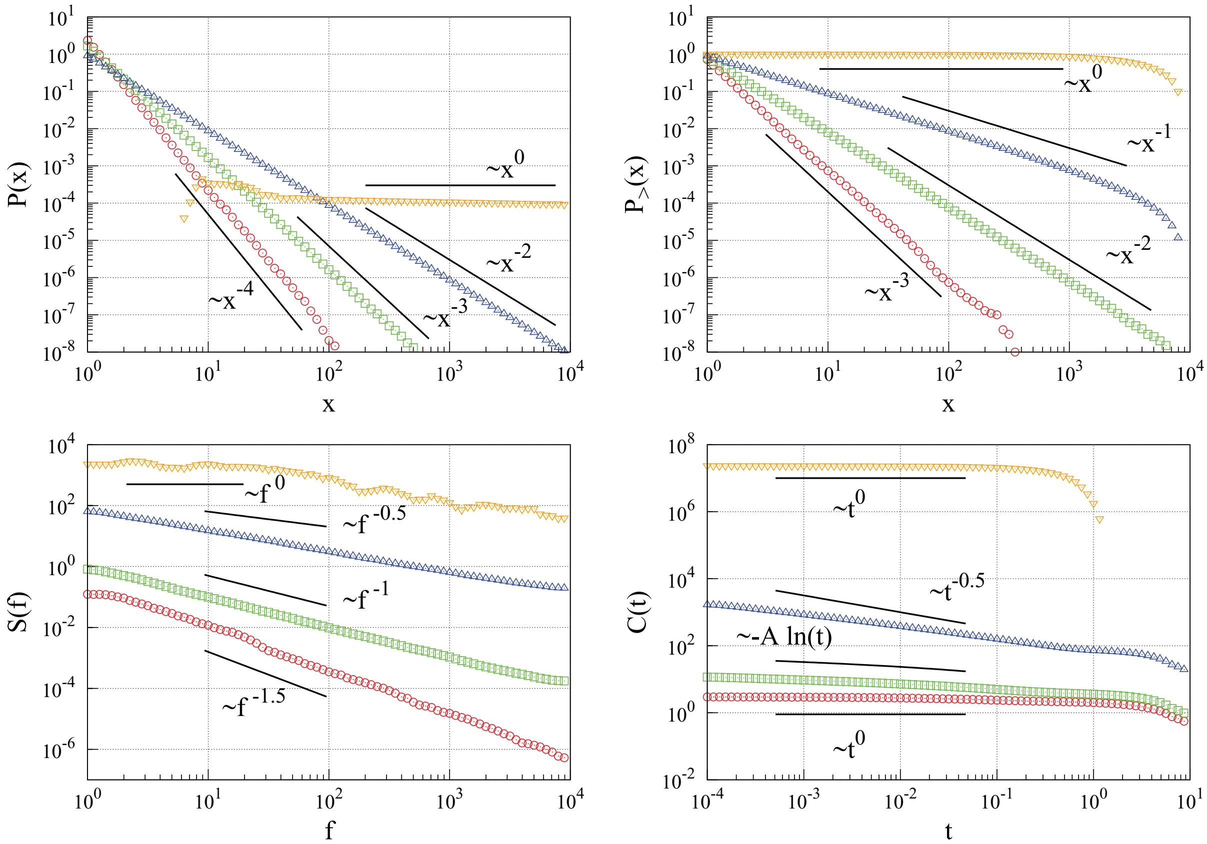

Figure 1. The steady-state probability distribution function $P(x)$, cumulative distribution $P_{>}(x)$, power spectral density $S(f)$, and autocorrelation function $C(t)$, of the variable x generated by equation (9) with restriction between $x_{\text{min}} = 1$ and $x_{\text{max}} = 10^4$; patterns from below are for different parameters $\alpha = 0$, $1/4$, $1/2$ and 1, respectively.

phenomena, while finite time studies reveal the main magnitudes of the observable (e.g. 'the inverse cubic law' in some finite interval). The very rare large deviations are hardly quantifiable and the tail-index estimation is challenging [50, 51].

Moreover, small fluctuations corresponding to the quiet market are relatively slow, the average delay time of system's feedback is relatively large in comparison to correlation time of the external weakly correlated rare influences, which can be represented by almost white noise. This process can be modeled by the (familiar in financial systems) Itô equations. On the other hand, the large rapid fluctuations of the violent market arise due to strong correlated influences (the herd behavior [47, 52–55]), the processes of such a market are fast, and all durations including the delay time of feedback are short in comparison to the herding correlation time. Consequently, the market should be modeled by the Stratonovich SDE.

To model these phenomena we generalize equations (1) and (9) with x-dependent parameter $\alpha(x)$,

$$\mathrm{d}x = 2\alpha(x)x^3\mathrm{d}t + x^2\mathrm{d}W_t. \tag{13}$$

with, e.g.

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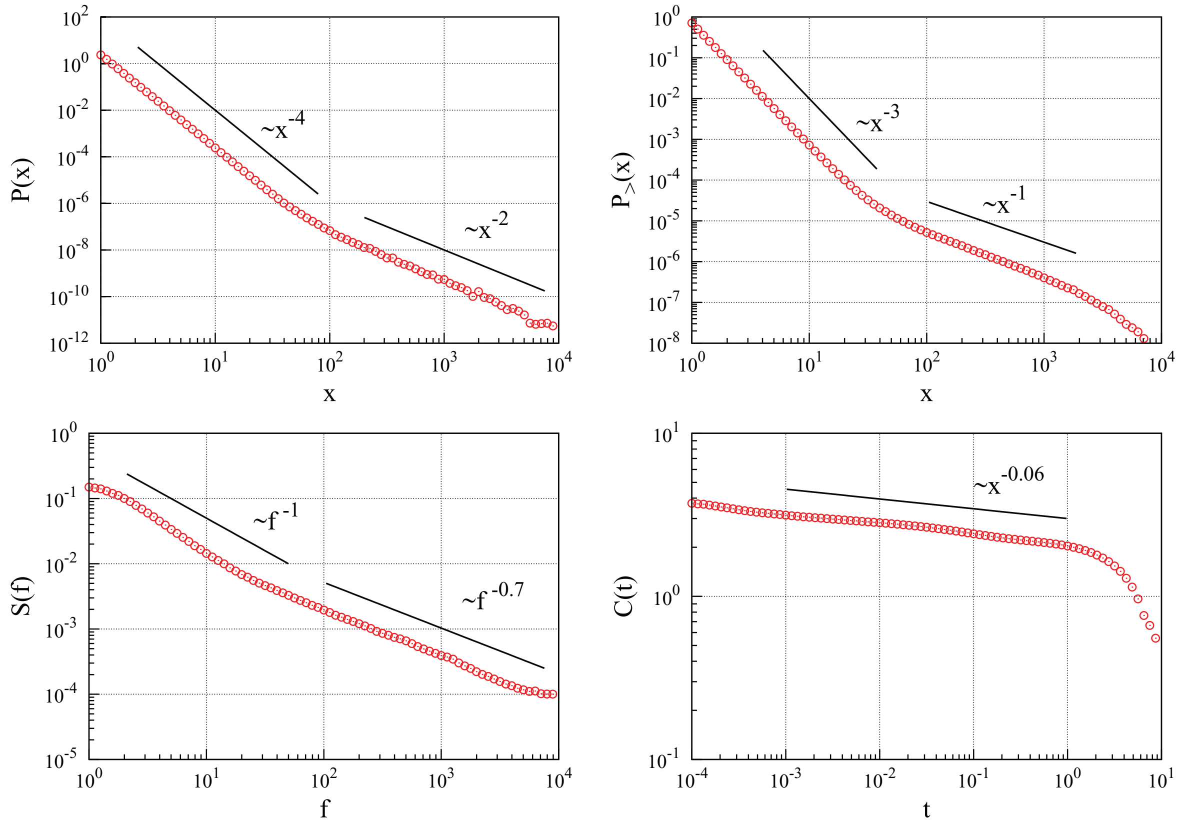

Figure 2. The steady-state probability distribution function $P(x)$, cumulative distribution $P_{>}(x)$, power spectral density $S(f)$, and autocorrelation function $C(t)$, generated by equations (13)–(16) with crossover value $x_c = 100$ and restrictions between $x_{\min} = 1$ and $x_{\max} = 10^4$.

$$\alpha(x) = \frac{1}{2}\left[1 - \exp\left\{-\left(\frac{x}{x_c}\right)^2\right\}\right], \tag{14}$$

where x_c is the Itô to Stratonovich interpretation crossover parameter. Equations (13) and (14) represent the transition from the Itô to Stratonovich convention with increasing variable x and decreasing delay time of multiplicative feedback for larger x, according to the Wong–Zakai theorem [31–35].

The calculations are performed with the variable step of integration

$$\Delta t_k = \kappa^2/x_k^2 \tag{15}$$

with $\kappa \ll 1$, yielding the difference equation

$$x_{k+1} = x_k + 2\kappa^2\alpha(x_k)x_k + \kappa x_k\varepsilon_k. \tag{16}$$

Here ε_k is a set of uncorrelated normally distributed random variables with zero expectation and unit variance.

Figure 2 demonstrates the results of the numerical calculations. Here we see the inverse cubic cumulative distribution, $P_{>}(x) \sim 1/x^3$, for $x \leqslant 50$, the power spectral density $1/f^{\beta}$ distribution with $\beta \approx 0.7$ for $f \geqslant 10$ and approximately $1/f$ noise for lower frequencies, similar to empirical data [14, 15, 25, 44]. The range of the inverse cubic distribution, $P_>(x) \sim 1/x^3$, and that of the $1/f^\beta$ spectrum with $\beta < 1$ may be increased to any desirable interval by increasing the crossover parameter x_c and the upper bound $x_{\max}$, respectively. Estimation of the tail index of very large and rare deviations is tricky [50, 51] and it cannot be deduced correctly. Therefore, these remote deviations can influence the spectrum but they are hardly observable and barely explorable directly. Different investigations result in different asymptotics of the heavy-tailed distributions, including log-normal, stretched-exponential, Weibull, incomplete Gamma and other distributions [17, 48, 56–59]. Therefore, the distribution $P(x) \sim x^2$ in some finite interval of the relatively large fluctuations as in our model, yielding $\beta < 1$ and long-range memory, does not contradict the known investigation outcomes.

5. Conclusions

We have modeled the long-range memory process with inverse cubic law of the cumulative distribution by a simple nonlinear SDE (1) with convention parameter $\alpha(x)$ dependent on the stochastic variable x. We have suggested that when the market evolves from calm to violent behavior there is a decrease of the delay time of multiplicative feedback of the system in comparison to the driving noise correlation time. This results in a transition from the Itô to the Stratonovich sense of the SDE (according to the Wong–Zakai theorem) and yields a long-range memory process. The transition from one to other sense of equation (13) is modeled by the dependence (14) of the convention parameter $\alpha(x)$ on the stochastic variable x. We have shown theoretically and numerically that SDE (1) with convention parameter $\alpha(x)$ (14) may reproduce the long memory inverse cubic phenomena.

It can be noted that equation (1) is very simple. All dependences on the parameters of the systems may be included by the appropriate scaling of the dimensionless variable x, dimensionless time t and the dimensionless crossover parameter x_c. The form of the dependence of the convention parameter $\alpha(x)$ on the stochastic variable x is not very essential and may differ from that represented by equation (14). Only the transition from α close to 0 for a calm market to $\alpha \approx 0.5$ for violent behavior is important.

In summary, the proposed simple stochastic model reproduces the well-known fact of the financial markets, i.e. the inverse cubic law of the cumulative distributions of the long-range memory fluctuations.

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