Physica A 411 (2014) 95–103

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Lévy flights in inhomogeneous environments and $1/f$ noise

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HIGHLIGHTS

- We obtained equations with Lévy stable noise and power law distribution of the signal.
- Gaussian noise can be replaced with Lévy stable noise preserving scaling of signal.
- For some parameters our equations generate signals having $1/f$ spectrum.

ARTICLE INFO

Article history:
Received 19 March 2014
Received in revised form 10 June 2014
Available online 20 June 2014

Keywords:
Stochastic analysis methods
Random walks and Lévy flights
Fractional equations
$1/f$ noise
Systems obeying scaling laws
Power law tails

ABSTRACT

Complex dynamical systems which are governed by anomalous diffusion often can be described by Langevin equations driven by Lévy stable noise. In this article we generalize nonlinear stochastic differential equations driven by Gaussian noise and generating signals with $1/f$ power spectral density by replacing the Gaussian noise with a more general Lévy stable noise. The equations with the Gaussian noise arise as a special case when the index of stability $\alpha = 2$. We expect that this generalization may be useful for describing $1/f$ fluctuations in the systems subjected to Lévy stable noise.

1. Introduction

The Lévy α-stable distributions, characterized by the index of stability $0 < \alpha \leqslant 2$, constitute the most general class of stable processes. The Gaussian distribution is their special case, corresponding to $\alpha = 2$. If $\alpha < 2$, then the Lévy stable distributions have power-law tails $\sim 1/x^{1+\alpha}$. There are many systems exhibiting Lévy α-stable distributions: distribution function of turbulent magnetized plasma emitters [1] and step-size distribution of photons in hot vapors of atoms [2] have Lévy tails; theoretical models suggest that velocity distribution of particles in fractal turbulence is Lévy stable distribution [3] or at least has Lévy tails [4]. If system behavior depends only on large noise fluctuations, such noise intensity distributions can be approximated by Lévy stable distribution, leading to Lévy flights. Lévy flight is a generalization of the Brownian motion which describes the motion of small macroscopic particles in liquid or gas experiencing unbalanced bombardments due to surrounding atoms. The Brownian motion mimics the influence of the "bath" on surrounding molecules in terms of time-dependent stochastic force which is commonly assumed to be white Gaussian noise. That postulate is compatible with the assumption of a short correlation time of fluctuations, much shorter than the time scale of the macroscopic motion, and the assumption of weak interactions with the bath. On the contrary, the Lévy motions describe the results of strong collisions between the particle and the surrounding environment. Lévy flights can be found in many physical systems: as an example we can point out anomalous diffusion of Na adatoms on a solid Cu surface [5], anomalous diffusion of a gold

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http://dx.doi.org/10.1016/j.physa.2014.06.020
0378-4371/

nanocrystal, adsorbed on the basal plane of graphite [6] and anomalous diffusion in optical lattices [7]. Lévy flights can be modeled by the fractional Fokker–Planck equations [8] or Langevin equations with additive Lévy stable noise.

Nonlinear stochastic differential equations (SDEs) with additive Lévy stable noise have been explored quite extensively for the past 15 years [9–12]. Such stochastic differential equations lead to the fractional Fokker–Planck equations with constant diffusion coefficient. Models with multiplicative Lévy stable noise have been used for modeling inhomogeneous media [13], ecological population density with fluctuating volume of resources [14]. The relation between the Langevin equation with multiplicative Lévy stable noise and the fractional Fokker–Planck equation has been introduced in Ref. [15], where the Langevin equation is interpreted in the Itô sense [16]. The relation between these two equations is not known in Stratonovich interpretation. Fractional Fokker–Planck equation models have been applied to model enzyme diffusion on polymer chain [17] and some cases of anomalous diffusion [18]. However, application of SDEs driven by Lévy stable noise can be problematic. We can always write the Fokker–Planck equation corresponding to the Langevin equation driven by the Gaussian noise and vice versa, but such statement is not always true for the Langevin equation with Lévy stable noise. For example, particle (enzyme) dispersion on a rapidly folding random heteropolymer can be described by the space fractional Fokker–Planck equation [19], but for such an equation the counterpart Langevin equation has not been found [20] and might not even exit [21].

One of the characteristics of the signal is the power spectral density (PSD). Signals having the PSD at low frequencies f of the form $S(f) \sim 1/f^{\beta}$ with β close to 1 are commonly referred to as "$1/f$ noise", "$1/f$ fluctuations", or "flicker noise". Such signals are often found in physics and in many other fields [22–28]. Since the discovery of $1/f$ noise, numerous models and theories have been proposed, for a recent review see Ref. [29]. However, most of those models and theories are not universal because they contain the assumptions specific to the problem under consideration. Mostly $1/f$ noise is considered as the Gaussian process [30,31], but sometimes the signal exhibiting $1/f$ fluctuations is non-Gaussian [32,33].

Often $1/f$ noise is modeled as the superposition of the Lorentzian spectra with a wide range distribution of relaxation times [34]. Self-organized criticality (SOC) [35–37] provides another important class of the models of $1/f$ noise. Yet another model has been presented in Refs. [38–41]: in this model the signals consist of pulses and it has been shown that a Brownian motion of the inter-pulse durations can yield $1/f$ noise. Starting from this model of $1/f$ noise nonlinear SDEs generating signals with $1/f$ spectrum were obtained in Refs. [42,43]. A special case of this nonlinear SDE has been obtained using Kirman's agent model [44]. Such nonlinear SDEs have been used to describe signals in socio-economical systems [45,46].

The purpose of this paper is to generalize nonlinear SDEs driven by the Gaussian noise and generate signals with $1/f$ PSD by replacing the Gaussian noise with a more general Lévy stable noise. The previously proposed SDEs then arise as a special case when $\alpha = 2$. We can expect that this generalization may be useful for describing $1/f$ fluctuations in the systems subject to Lévy stable noise.

The paper is organized as follows: In Section 2 we search for the nonlinear SDE with Lévy stable noise yielding power law steady state probability density function (PDF) of the generated signal. In Section 3 we estimate when the signal generated by such an SDE has $1/f$ PSD in a wide region of frequencies. In Section 4 we numerically solve the obtained equations and compare the PDF and PSD of the signal with analytical estimations. Section 5 summarizes our findings.

2. Stochastic differential equation with Lévy stable noise generating signals with power law distribution

In this section we search for nonlinear SDEs with Lévy stable noise yielding power law steady state PDF of the generated signal. We consider the Langevin equation of the form [8,47,48]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(x) + b(x)\xi(t), \tag{1}$$

where $a(x)$ and $b(x)$ are the given functions describing the deterministic drift term and the amplitude of the noise, respectively. The stochastic force $\xi(t)$ is uncorrelated, $\langle \xi(t)\xi(t')\rangle = \delta(t - t')$ and is characterized by Lévy α-stable distribution. In this paper we will restrict our investigation only to symmetric stable distributions, thus the characteristic function of $\xi(t)$ is

$$\langle \exp(\mathrm{i}k\xi)\rangle = \exp(-\sigma^{\alpha}|k|^{\alpha}). \tag{2}$$

Here α is the index of stability and σ is the scale parameter. We interpret Eq. (1) in the Itô sense. In a mathematically more formal way Eq. (1) can be written in the form

$$\mathrm{d}x = a(x)\mathrm{d}t + b(x)\mathrm{d}L_t^{\alpha}, \tag{3}$$

where $\mathrm{d}L_t^{\alpha}$ stands for the increments of Lévy α-stable motion L_t^{α} [49,50]. For calculating the steady state PDF of the signal x we will use the fractional Fokker–Planck equation instead of the stochastic differential equation (1). The fractional Fokker–Planck equation corresponding to the Itô solution of Eq. (1) is [51,15]

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}a(x)P(x,t) + \sigma^{\alpha}\frac{\partial^{\alpha}}{\partial |x|^{\alpha}}b(x)^{\alpha}P(x,t). \tag{4}$$

Here $\partial^\alpha/\partial|x|^\alpha$ is the Riesz–Weyl fractional derivative. The Riesz–Weyl fractional derivative of the function $f(x)$ is defined by its Fourier transform [52],

$$\mathcal{F}\left[\frac{\partial^\alpha}{\partial|x|^\alpha}f(x)\right]=-|k|^\alpha\tilde{f}(k). \tag{5}$$

One can get the following expression for the Riesz–Weyl derivative:

$$\frac{\partial^\alpha}{\partial|x|^\alpha}f(x)=-\frac{1}{2\cos\left(\frac{\pi\alpha}{2}\right)}\{D_+^{-\alpha}f(x)+D_-^{-\alpha}f(x)\}, \tag{6}$$

where $D_+^{-\alpha}$ and $D_-^{-\alpha}$ are the left and right Riemann–Liouville derivatives [52]:

$$D_\pm^{-\alpha}=(\pm 1)^m\frac{\mathrm{d}^m}{\mathrm{d}x^m}D_\pm^{m-\alpha},\quad m-1<\alpha<m. \tag{7}$$

Here m is an integer and

$$D_+^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-z)^{\alpha-1}f(z)\,\mathrm{d}z, \tag{8}$$

$$D_-^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{+\infty}(z-x)^{\alpha-1}f(z)\,\mathrm{d}z. \tag{9}$$

When $\alpha=1$ then the definition of the Riesz–Weyl derivative is

$$\frac{\mathrm{d}}{\mathrm{d}|x|}f(x)=-\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{f(z)}{x-z}\,\mathrm{d}z. \tag{10}$$

Eq. (4) leads to the following equation for the steady state PDF:

$$\sigma^\alpha\frac{\partial^\alpha}{\partial|x|^\alpha}b(x)^\alpha P_0(x)-\frac{\partial}{\partial x}a(x)P_0(x)=0. \tag{11}$$

Eq. (11) can be written as $-\mathrm{d}J(x)/\mathrm{d}x=0$, where $J(x)$ is the probability current. Reflective boundaries lead to the boundary condition $J(x)=0$.

2.1. Equation with only positive values of x

We will search for the stochastic differential equation (1) generating signals with power law steady state PDF,

$$P_0(x)\sim x^{-\lambda}. \tag{12}$$

Since the power law PDF cannot be normalized when x can vary from zero to infinity, we will assume that the power law holds only in some wide regions of x, $x_{\min}\ll x\ll x_{\max}$. One can expect that the power law PDF can be obtained when the coefficients $a(x)$ and $b(x)$ in Eq. (1) themselves are of the power law form. Thus we will consider $b(x)=x^\eta$ and $a(x)=\sigma^\alpha\gamma x^\mu$. Here η is the exponent of the multiplicative noise, μ and γ must be determined. With such a choice of $b(x)$ and power law form of $P_0(x)$ from Eq. (4) it follows that we need to calculate the fractional derivative of the power law function.

Let us consider the function

$$f(x)=\begin{cases}x^\rho, & x_{\min}<x<x_{\max},\\ 0 & \text{otherwise.}\end{cases} \tag{13}$$

Using Eq. (6) we obtain the following approximate expressions for the fractional derivative of the function (13) when $x_{\min}\ll x\ll x_{\max}$:

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}|x|^\alpha}f(x)\approx\begin{cases}\dfrac{\sin\left(\pi\left(\frac{\alpha}{2}-\rho\right)\right)}{\sin(\pi(\rho-\alpha))}\dfrac{\Gamma(1+\rho)}{\Gamma(1+\rho-\alpha)}x^{\rho-\alpha}, & -1<\rho<\alpha,\\[2ex] \dfrac{x_{\min}^{1+\rho}}{2\cos\left(\frac{\pi}{2}\alpha\right)(1+\rho)\Gamma(-\alpha)}x^{-1-\alpha}, & \rho<-1 \qquad 0<\alpha<2;\ \alpha\neq 1\\[2ex] \dfrac{x_{\max}^{\rho-\alpha}}{2\cos\left(\frac{\pi}{2}\alpha\right)(\alpha-\rho)\Gamma(-\alpha)}, & \rho>\alpha,\end{cases} \tag{14}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}|x|}f(x) \approx \begin{cases} -\lambda\cot(\pi\rho)x^{\rho-1}, & -1<\rho<1 \\ -\frac{x_{\min}^{1+\rho}}{\pi(1+\rho)}x^{-2}, & \rho<-1 \\ \frac{x_{\max}^{\rho-1}}{\pi(1-\rho)}, & \rho>1 \end{cases} \tag{15}$$

for $\alpha = 1$. We see that the approximate expression for the fractional derivative does not depend on the limiting values $x_{\min}$ and $x_{\max}$ when $-1 < \rho < \alpha$. Using the power-law forms of the coefficients $a(x)$ and $b(x)$, assuming that $-1 < \alpha\eta - \lambda < \alpha$ and using Eq. (14) for the fractional derivative, from Eq. (11) we get

$$\frac{\sin\left[\pi\left(\frac{\alpha}{2}-\alpha\eta+\lambda\right)\right]}{\sin[\pi(\alpha(\eta-1)-\lambda)]}\frac{\Gamma(1+\alpha\eta-\lambda)}{\Gamma(1+\alpha(\eta-1)-\lambda)}x^{\alpha(\eta-1)-\lambda} - \gamma(\mu-\lambda)x^{\mu-\lambda-1} = 0. \tag{16}$$

This equation should be valid for all values of x. This can be only when

$$\mu = \alpha(\eta-1)+1 \tag{17}$$

and

$$\gamma = \frac{\sin\left[\pi\left(\frac{\alpha}{2}-\alpha\eta+\lambda\right)\right]}{\sin[\pi(\alpha(\eta-1)-\lambda)]}\frac{\Gamma(\alpha\eta-\lambda+1)}{\Gamma(\alpha(\eta-1)-\lambda+2)}. \tag{18}$$

Thus we will investigate the nonlinear SDE with Lévy stable noise of the form

$$\mathrm{d}x = \sigma^{\alpha}\frac{\sin\left[\pi\left(\frac{\alpha}{2}-\alpha\eta+\lambda\right)\right]}{\sin[\pi(\alpha(\eta-1)-\lambda)]}\frac{\Gamma(\alpha\eta-\lambda+1)}{\Gamma(\alpha(\eta-1)-\lambda+2)}x^{\alpha(\eta-1)+1}\mathrm{d}t + x^{\eta}\mathrm{d}L_t^{\alpha}. \tag{19}$$

This equation is a generalization of the nonlinear SDE with Gaussian noise proposed in Refs. [42,43]. Because of the divergence of the power law distribution and the requirement of the stationarity of the process, SDE (19) should be analyzed together with the appropriate restrictions of the diffusion in some finite interval. The simplest choice of restriction is the reflective boundaries at $x = x_{\min}$ and $x = x_{\max}$. However, other forms of restrictions are possible by introducing additional terms in the drift term of Eq. (19).

From Eq. (14) it follows that the equation for the fractional derivative is valid when $-1 < \alpha\eta - \lambda < \alpha$. However, the condition $J(x) = 0$ for the probability current leads to a stronger restriction than Eq. (11) which ensures only $\mathrm{d}J(x)/\mathrm{d}x = 0$. Using Eq. (6) and the function (13) we see that the upper limiting value $x_{\max}$ can be neglected in the probability current when $\rho < \alpha - 1$. Thus the power law exponent λ of the steady state PDF should be from the interval

$$\alpha(\eta-1)+1 < \lambda < \alpha\eta+1. \tag{20}$$

As a particular case when $\alpha = 2$ from Eq. (19) we get the previously proposed SDE with the Gaussian noise [42,43]

$$\mathrm{d}x = \sigma^2(2\eta-\lambda)x^{2\eta-1}\mathrm{d}t + x^{\eta}\mathrm{d}L_t^2. \tag{21}$$

Note, that according to the definition (2), the scale parameter σ differs from the standard deviation of the Gaussian noise. Eq. (19) has a simple form when $\alpha = 1$:

$$\mathrm{d}x = \sigma\cot[\pi(\lambda-\eta)]x^{\eta}\mathrm{d}t + x^{\eta}\mathrm{d}L_t^1. \tag{22}$$

2.2. Equations allowing both positive and negative values of x

In Eq. (19) the stochastic variable x can acquire only positive values. Similarly as in Ref. [53] we can get the equations allowing x to be negative. We will search for the stochastic differential equation (1) generating signals with power law steady state PDF

$$P_0(x) \sim |x|^{-\lambda}. \tag{23}$$

To have a normalizable PDF we will assume that the power law holds only in some wide regions of x, $x_{\min} \ll |x| \ll x_{\max}$. In order to obtain such an equation we will consider Eq. (1) with the coefficients having the power law form $a(x) = \sigma^{\alpha}\gamma|x|^{\mu-1}x$ and $b(x) = |x|^{\eta}$ when $|x| \gg x_{\min}$. Similarly as in the case of the positive x we investigate the fractional derivative of the function

$$f(x) = \begin{cases} |x|^{\rho}, & x_{\min} < |x| < x_{\max}, \\ x_{\min}^{\rho}, & -x_{\min} < x < x_{\min}, \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

Using Eq. (6) we obtain the following approximate expressions for the fractional derivative of the function (13) when $x_{\min} \ll x \ll x_{\max}$:

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}|x|^{\alpha}} f(x) \approx \frac{\sin\left(\frac{\pi}{2}\rho\right)}{\sin\left(\frac{\pi}{2}(\alpha-\rho)\right)} \frac{\Gamma(1+\rho)}{\Gamma(1+\rho-\alpha)} x^{\rho-\alpha}, \quad -1 < \rho < \alpha. \tag{25}$$

Using Eq. (25) for the fractional derivative in Eq. (11), we obtain $\mu = \alpha(\eta - 1) + 1$ and

$$\gamma = \frac{\sin\left[\frac{\pi}{2}(\alpha\eta-\lambda)\right]}{\sin\left[\frac{\pi}{2}(\lambda-\alpha(\eta-1))\right]} \frac{\Gamma(\alpha\eta-\lambda+1)}{\Gamma(\alpha(\eta-1)-\lambda+2)}. \tag{26}$$

In addition, from Eq. (25) it follows that the power law exponent λ of the steady state PDF should be from the interval

$$\alpha(\eta - 1) < \lambda < \alpha\eta + 1. \tag{27}$$

When $\alpha = 2$, Eq. (26) simplifies to

$$\gamma = 2\eta - \lambda. \tag{28}$$

This expression is the same as the one for the SDE with only positive values of x and $\alpha = 2$. However, when $\alpha < 2$, the coefficient γ given by Eq. (26) is different from γ given by Eq. (18), in contrast to the Gaussian case ($\alpha = 2$). This can be understood by noticing that the Lévy stable noise for $\alpha < 2$ has large jumps. Jumps from the regions with negative values of the stochastic variable x to the regions with positive values influence the PDF $P_0(x)$ for the positive values of x. The situation is the same with the jumps from positive to negative regions. Eq. (26) also has a simple form

$$\gamma = \tan\left[\frac{\pi}{2}(\eta - \lambda)\right] \tag{29}$$

for $\alpha = 1$.

The required form of the coefficients $a(x)$ and $b(x)$ has the equation

$$\mathrm{d}x = \sigma^{\alpha}\gamma(x_0^2 + x^2)^{\frac{\alpha}{2}(\eta-1)} x\mathrm{d}t + (x_0^2 + x^2)^{\frac{\eta}{2}} \mathrm{d}L_t^{\alpha} \tag{30}$$

and the equation

$$\mathrm{d}x = \sigma^{\alpha}\gamma(x_0^{\alpha} + |x|^{\alpha})^{\eta-1} x\mathrm{d}t + (x_0^{\alpha} + |x|^{\alpha})^{\frac{\eta}{\alpha}} \mathrm{d}L_t^{\alpha}. \tag{31}$$

Here parameter x_0 plays the role of $x_{\min}$. The restriction of the diffusion at the large absolute values of x can be achieved by reflective boundaries at $\pm x_{\max}$ or by additional terms in the equations. Eq. (30) is a generalization of SDE with Gaussian noise from Ref. [53]. The addition of the parameter x_0 restricts the divergence of the power law distribution of x at $x \to 0$. Eqs. (30) and (31) for $|x| \ll x_0$ represent SDEs with additive Lévy stable noise and linear relaxation.

3. Power spectral density of the generated signals

In this section we estimate the PSD of the signals generated by the SDE with Lévy stable noise

$$\mathrm{d}x = \sigma^{\alpha}\gamma x^{\alpha(\eta-1)+1}\mathrm{d}t + x^{\eta}\mathrm{d}L_t^{\alpha}, \tag{32}$$

proposed in the previous section. Here γ is given by Eq. (18). For this estimation we use the (approximate) scaling properties of the signals, as it is done in Appendix A of Ref. [54] and in Ref. [55]. Using the Wiener–Khintchine theorem the PSD can be related to the autocorrelation function $C(t)$, which can be calculated using the steady state PDF $P_0(x)$ and the transition probability $P(x', t|x, 0)$ (the conditional probability that at time t the signal has value x' with the condition that at time $t = 0$ the signal had the value x) [56]:

$$C(t) = \int \mathrm{d}x \int \mathrm{d}x'\, xx' P_0(x) P(x', t|x, 0). \tag{33}$$

The transition probability can be obtained from the solution of the fractional Fokker–Planck equation (4) with the initial condition $P(x', t = 0|x, 0) = \delta(x' - x)$.

The increments of Lévy α-stable motion $\mathrm{d}L_t^{\alpha}$ have the scaling property $\mathrm{d}L_{at}^{\alpha} = a^{1/\alpha}\mathrm{d}L_t^{\alpha}$ [49]. Changing the variable x in Eq. (32) to the scaled variable $x_s = ax$ or introducing the scaled time $t_s = a^{\alpha(\eta-1)}t$ one gets the same resulting equation. Thus the change of the scale of the variable x and change of time scale are equivalent, leading to the following scaling property of the transition probability:

$$aP(ax', t|ax, 0) = P(x', a^{\mu}t|x, 0), \tag{34}$$

with the exponent μ being

$$\mu = \alpha(\eta - 1). \tag{35}$$

As has been shown in Ref. [55], the power law steady state PDF $P_0(x) \sim x^{-\lambda}$ and the scaling property of the transition probability (34) lead to the power law form PSD $S(f) \sim f^{-\beta}$ in a wide range of frequencies. From the equation

$$\beta = 1 + (\lambda - 3)/\mu, \tag{36}$$

obtained in Ref. [55], it follows that the power-law exponent in the PSD of the signal generated by SDE with Lévy stable noise (32) is

$$\beta = 1 + \frac{\lambda - 3}{\alpha(\eta - 1)}. \tag{37}$$

This expression is the generalization of the expression for the power-law exponent in the PSD with $\alpha = 2$, obtained in Ref. [43]. As Eq. (37) shows, we get $1/f$ PSD when $\lambda = 3$.

The presence of the restrictions at $x = x_{\min}$ and $x = x_{\max}$ makes the scaling (34) not exact and this limits the power law part of the PSD to a finite range of frequencies $f_{\min} \ll f \ll f_{\max}$. Similarly as in Ref. [55] we can estimate the limiting frequencies. Taking into account $x_{\min}$ and $x_{\max}$ the autocorrelation function has the scaling property [55]

$$C(t; ax_{\min}, ax_{\max}) = a^2 C(a^{\mu} t, x_{\min}, x_{\max}).$$

This equation means that time t in the autocorrelation function should enter only in combinations with the limiting values, $x_{\min} t^{1/\mu}$ and $x_{\max} t^{1/\mu}$. We can expect that the influence of the limiting values can be neglected when the first combination is small and the second large, that is when time t is in the interval $\sigma^{-\alpha} x_{\max}^{\alpha(1-\eta)} \ll t \ll \sigma^{-\alpha} x_{\min}^{\alpha(1-\eta)}$. Then the frequency range where the PSD has $1/f^{\beta}$ behavior can be estimated as

$$\sigma^{\alpha} x_{\min}^{\alpha(\eta-1)} \ll 2\pi f \ll \sigma^{\alpha} x_{\max}^{\alpha(\eta-1)}. \tag{38}$$

This equation shows that the frequency range grows with increasing of the exponent η, the frequency range becomes zero when $\eta = 1$. By increasing the ratio $x_{\max}/x_{\min}$ one can get arbitrarily wide range of the frequencies where the PSD has $1/f^{\beta}$ behavior. Note, that pure $1/f^{\beta}$ PSD is physically impossible because the total power would be infinite. Therefore, we consider signals with PSD having $1/f^{\beta}$ behavior only in some wide intermediate regions of frequencies, $f_{\min} \ll f \ll f_{\max}$, whereas for small frequencies $f \ll f_{\min}$ PSD is bounded.

The numerical solution of the SDEs with Lévy noise shows that the frequency range where the PSD has $1/f^{\beta}$ behavior falls within the interval determined by Eq. (38) and this equation gives qualitatively correct dependence of the limiting frequencies on the parameters. However, the limiting frequencies obtained from numerical solution indicate a narrower frequency range: the minimum limiting frequency $f_{\min}$ differs from that given by Eq. (38) not more than by a factor of 10, whereas the maximum limiting frequency $f_{\max}$ is much smaller than in Eq. (38). In addition, numerically obtained limiting frequencies depend also on the parameter λ. This discrepancy arises because the reasoning leading to Eq. (38) is not strict and has more heuristic nature. To obtain more correct limiting frequencies it is insufficient to consider only the approximate scaling properties of the nonlinear SDE.

The power spectral density of the form $1/f^{\beta}$ is determined mainly by the power law behavior of the coefficients of SDE (32) at large values of $x \gg x_{\min}$. Changing the coefficients at small x, the spectrum preserves the power law behavior. The modifications of SDE (30), (31) and the introduction of negative values of the stochastic variable x should not destroy the frequency region with $1/f^{\beta}$ behavior of the power spectral density. This is confirmed by the numerical solution of the equations.

4. Numerical examples

When $\lambda = 3$, we get that $\beta = 1$ and SDEs (19), (30), (31) should give a signal exhibiting $1/f$ noise. We will solve numerically two cases, corresponding to Eqs. (19) and (30), with the index of stability of Lévy stable noise $\alpha = 1$ and the power law exponent of the steady state PDF $\lambda = 3$. Note, that for this value of α the Lévy α-stable distribution is the same as the Cauchy distribution. For simplicity we choose the exponent in the noise amplitude η such that the coefficient γ, given by Eq. (18) or (26), becomes equal to -1. For the numerical solution we use Euler's approximation, transforming differential equations to difference equations. Eq. (32) leads to the following difference equation:

$$x_{k+1} = x_k + \sigma^{\alpha} \gamma x_k^{\alpha(\eta-1)+1} h_k + x_k^{\eta} h_k^{1/\alpha} \xi_k^{\alpha}, \tag{39}$$

where $h_k = t_{k+1} - t_k$ is the time step and ξ_k^{α} is a random variable having α-stable Lévy distribution with the characteristic function (2). We can solve Eq. (39) numerically with the constant step $h_k = \text{const}$. However, a more effective method of the solution of Eq. (39) is when the change of the variable x_k in one step is proportional to the value of the variable, as has been done solving SDE with Gaussian noise in Ref. [42]. The variable step of integration

$$h_k = \frac{\kappa^{\alpha}}{\sigma^{\alpha}} x_k^{-\alpha(\eta-1)} \tag{40}$$

results in the equation

$$x_{k+1} = x_k + \kappa^{\alpha} \gamma x_k + \frac{\kappa}{\sigma} x_k \xi_k^{\alpha}. \tag{41}$$

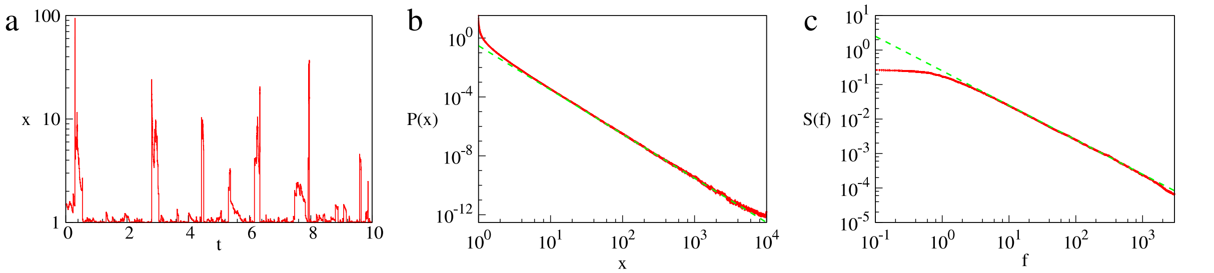

Fig. 1. (a) Signal generated by SDE with Lévy stable noise (42) with reflective boundaries at $x = x_{\min}$ and $x = x_{\max}$. (b) Steady state PDF $P_0(x)$ of the signal. The dashed green line shows the slope x^{-3}. (c) Power spectral density $S(f)$ of the signal. The dashed green line shows the slope $1/f$. Parameters used are $x_{\min} = 1$, $x_{\max} = 10^4$, $\sigma = 1$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Here $\kappa \ll 1$ is a small parameter. We include the reflective boundaries at $x = x_{\min}$ and $x = x_{\max}$ using the projection method [57,58]. According to the projection method, if the variable x_{k+1} acquires the value outside the interval $[x_{\min}, x_{\max}]$, then the value of the nearest reflective boundary is assigned to x_{k+1}.

When $\alpha = 1$, $\lambda = 3$ and $\eta = 9/4$, SDE (19) is

$$dx = \sigma x^{9/4} dt + x^{9/4} dL_t^1. \tag{42}$$

The results obtained numerically solving this equation with reflective boundaries at $x = x_{\min}$ and $x = x_{\max}$ are shown in Fig. 1. A sample of the generated signal is shown in Fig. 1(a). The signal exhibits peaks or bursts, corresponding to the large deviations of the variable x. Comparison of the steady state PDF $P_0(x)$ and the PSD $S(f)$ with the analytical estimations is presented in Fig. 1(b) and (c). There is quite good agreement of the numerical results with the analytical expressions. In Fig. 1(b) we see that near the reflecting boundaries the steady state PDF deviates from the power law prediction. This increase of the steady state PDF near boundaries is typical for equations with Lévy stable noise having $\alpha < 2$ [12]. The behavior of the steady state PDF near the reflecting boundaries is similar to the behavior of the analytical expression obtained in Ref. [12] for the simplest stochastic differential equation Lévy stable noise having constant noise amplitude and zero drift.

A numerical solution of the equations confirms the presence of the frequency region for which the PSD has $1/f$ dependence. The width of this region can be increased by increasing the ratio between the minimum and the maximum values of the stochastic variable x. In addition, the region in the PSD with the power law behavior depends on α and the exponent η: the width increases with increasing the difference $\eta - 1$ and increasing α; when $\eta = 1$ then this width is zero. Such behavior is correctly predicted by Eq. (38).

We use similar schemes of numerical solution also for SDEs (30) and (31). Euler's approximation with variable step of integration

$$h_k = \frac{\kappa^\alpha}{\sigma^\alpha} (x_0^2 + x_k^2)^{-\frac{\alpha}{2}(\eta-1)} \tag{43}$$

transforms SDE (30) to the difference equation

$$x_{k+1} = x_k + \kappa^\alpha \gamma x_k + \frac{\kappa}{\sigma} \sqrt{x_0^2 + x_k^2}\, \xi_k^\alpha. \tag{44}$$

For SDE (31) we use the variable step of integration

$$h_k = \frac{\kappa^\alpha}{\sigma^\alpha} (x_0^\alpha + |x_k|^\alpha)^{-(\eta-1)} \tag{45}$$

resulting in the difference equation

$$x_{k+1} = x_k + \kappa^\alpha \gamma x_k + \frac{\kappa}{\sigma} (x_0^\alpha + |x_k|^\alpha)^{\frac{1}{\alpha}} \xi_k^\alpha. \tag{46}$$

Here $\kappa \ll 1$ is a small parameter. We include reflective boundaries at $x = \pm x_{\max}$ using the projection method.

When $\alpha = 1$, $\lambda = 3$ and $\eta = 5/2$, the SDE (30) with the coefficient γ given by Eq. (26) is

$$dx = -\sigma (x_0^2 + x^2)^{3/4} x dt + (x_0^2 + x^2)^{5/4} dL_t^1. \tag{47}$$

The results obtained numerically solving this equation with reflective boundaries at $x = \pm x_{\max}$ are shown in Fig. 2. A sample of the generated signal is shown in Fig. 2(a). Comparison of the steady state PDF $P_0(x)$ and the PSD $S(f)$ with the analytical estimations is presented in Fig. 2(b) and (c). There is quite good agreement of the numerical results with the analytical expressions. As in the case with only positive values of x, we see in Fig. 1(b) we see the increase of the steady state PDF near the reflecting boundaries $x = \pm x_{\max}$ in comparison to the power law prediction. Numerical solution of Eq. (47) confirms the presence of the frequency region where the PSD has $1/f^\beta$ dependence.

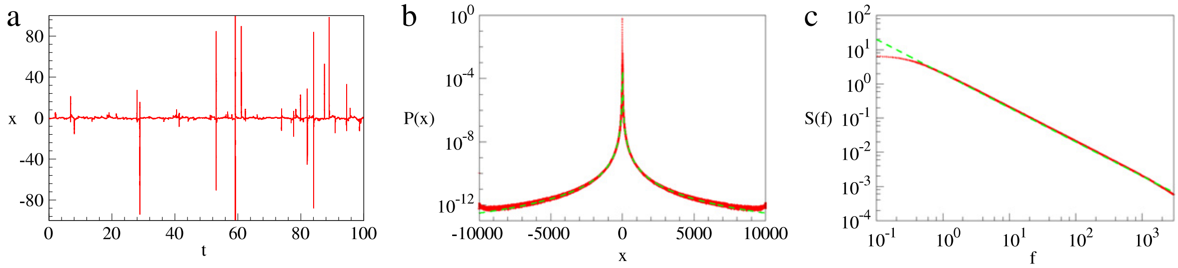

Fig. 2. (a) Signal generated by SDE with Lévy stable noise (47). (b) Steady state PDF $P_0(x)$ of the signal. The dashed green line shows the dependence on x proportional to $|x|^{-3}$. (c) Power spectral density $S(f)$ of the signal. The dashed green line shows the slope $1/f$. Parameters used are $x_0 = 1$, $x_{\max} = 10^4$, $\sigma = 1$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

5. Discussion

Lévy flights have been modeled using the Langevin equation with various subharmonic potentials and additive Lévy stable noise [9,20,10,21]. Proposed SDE (19) contains multiplicative Lévy stable noise and is a generalization of previous attempts to model Lévy flights. This SDE can be used to investigate Lévy flights in non-equilibrium and non-homogeneous environments, like porous media and some cases of polymer chains [19,17]. If specific conditions given by Eq. (20) are satisfied, our model generates Lévy flights exhibiting $1/f$ noise. The drift term $a(x)$ in Eq. (19) represents a subharmonic external force effecting the particle. Lévy flights in subharmonic potentials lead to various interesting phenomena such as stochastic resonance in single well potential [59]. The power law dependence of the diffusion coefficient $b^2(x)$ on the stochastic variable x can be traced to the existence of the energy flux due to temperature gradient in a bath. Long jumps leading to Lévy stable noise can arise from a complex scale free structure of the bath as is in the case of enzyme diffusion on a polymer [19]. There are suggestions that the non-homogeneity of the bath can be described by the dependence of the diffusion coefficient on the particle coordinate x [13] and Lévy stable noise arises from the bath not being in an equilibrium.

In the case of Gaussian noise ($\alpha = 2$) the nonlinear SDE (19) that generates the signal with $1/f$ spectrum can be obtained from various models. One of those models is a signal consisting from a sequence of pulses with a Brownian motion of the inter-pulse durations [42,43]. This suggests that our more general form of the SDE could be obtained from some kind of Lévy motion of the inter-pulse durations. However, we were unable to show this due to the complexity of the Itô formula in case of equations driven by Lévy process [60]. The special case of Eq. (19) for a free particle ($a(x) = 0$) with Lévy stable noise having $\alpha < 2$ has been derived from coupled continuous time random walk (CTRW) models [18], when the jumping rate ν of CTRW process depends on signal intensity as $\nu(x) = x^{\alpha\eta}$, $x > 0$. However, such derivation is quite complex and does little to help understanding what kind of physical phenomena can be approximated by multiplicative Lévy stable noise. Thus instead of searching for underlying models in this article we have chosen a simpler approach: we have derived nonlinear SDEs using a simple reasoning about scaling properties of the steady state PDF.

Taking into account the scaling properties of the signal is one of the advantages of our model, in many theoretical models, such as diffusion of the particle in a fractal turbulence [3], ecological population density with fluctuating volume of resources [14], dynamics of two competing species [61] and tumor growth [62], the existence of Lévy stable noise instead of Gaussian noise is simply assumed. Such an assumption might be incorrect, because the change of statistical properties of the noise changes the scaling properties of the signal. In order to preserve original scaling properties of the signal the drift $a(x)$ or diffusion $b^2(x)$ coefficients must be changed as well. The required drift coefficient $a(x)$ can be found similarly as in Section 2. The scaling properties can be extracted from time series using fluctuation analysis methods [50].

In summary, we have proposed nonlinear SDEs with Lévy stable noise and generating signals exhibiting $1/f$ noise in any desirably wide range of frequency. Proposed SDEs (19), (30) and (31) are a generalization of nonlinear SDEs driven by Gaussian noise and generating signals with $1/f$ PSD. The generalized equations can be obtained by replacing the Gaussian noise with the Lévy stable noise and changing the drift term to preserve statistical properties of the generated signal. We have investigated two cases: in the first case the stochastic variable can acquire only positive values (SDE (19)), in the second case the stochastic variable can also be negative (SDEs (30) and (31)). In contrast to the SDEs with the Gaussian noise, the constant in the drift term, given by Eqs. (18) and (26), is different in those two cases and becomes the same only for $\alpha = 2$.

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