



Order book model with herd behavior exhibiting long-range memory

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HIGHLIGHTS

- The model is based on empirical order book model and financial herd behavior model.
- The model is compared against high frequency data from Bitcoin exchanges and NYSE.
- Fracture in PSD can be explained by the convergence towards the Walras equilibrium.

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ABSTRACT

In this work, we propose an order book model with herd behavior. The proposed model is built upon two distinct approaches: a recent empirical study of the detailed order book records by Kanazawa et al. (2018) and financial herd behavior model. Combining these approaches allows us to propose a model that replicates the long-range memory of absolute return and trading activity. We compare the statistical properties of the model against the empirical statistical properties of the Bitcoin exchange rates and New York stock exchange tickers. We also show that the fracture in the spectral density of the high-frequency absolute return time series might be related to the mechanism of convergence towards the equilibrium price.

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1. Introduction

In the recent decades an increasing effort by social scientists, physicists and broader interdisciplinary community has been applied to create agent-based models (ABMs) of different social phenomena [1–8]. Notable part of these ABMs were created to explain various recurrent anomalous statistical patterns, collectively referred to as stylized facts, observed in the financial markets [2–4,7]. Financial ABMs vary in complexity usually trading plausibility for analytical tractability [4]. Some of the more complex financial ABMs seem to be reasonably plausible, but they are not analytically tractable. A well known example of such model is the Lux–Marchesi model [9]. Hence it significantly harder to understand their dynamics and to build upon them to further improve their agreement with the empirical data.

In the recent few years we have developed a reasonably plausible yet analytically tractable financial ABM [10,11]. This financial ABM was derived from a widely recognized behavioral model [12], which emphasizes imitation (herd) behavior among socially interacting individuals. This financial ABM is able to rather precisely fit the probability density functions (PDFs) and power spectral densities (PSDs) of the empirical absolute return time series. While the model has other desirable features, e.g., it scales well with change in the time scale, it also has some drawbacks. First of all it does not implement realistic trading strategies, though most of the financial ABMs lack this feature [4]. But it is not the main

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Table 1

Comparison of the approaches reviewed in Section 2. Differences between our approach and the reviewed approaches are highlighted using *italics*.

Approach	Agent types	Fixed/flexible	Strategies	Volume	Tractable
[27]	Chartists, fundamentalists	<i>Fixed</i>	<i>Sound</i>	<i>Strategic</i>	Partly analytically
[28]	Chartists, fundamentalists	<i>Fixed</i>	Simple	Unit	<i>Only numerically</i>
[29–31]	Chartists, fundamentalists	<i>Fixed</i>	<i>Sound</i>	<i>Random</i>	<i>Only numerically</i>
[32]	Chartists, fundamentalists	<i>Fixed</i>	<i>Realistic</i>	Unit	<i>Only numerically</i>
[33,34]	Chartists, <i>random traders</i>	<i>Fixed</i>	Simple	<i>Random</i>	<i>only numerically</i>
[35]	Chartists, fundamentalists	<i>Fixed</i>	<i>Realistic</i>	<i>Strategic</i>	<i>Only numerically</i>
Our	Chartists, fundamentalists	Flexible	Simple	Unit	Possibly analytically

issue we currently see. Our financial ABM relies on a two assumptions, which are not fully and transparently justified. We have assumed the presence of the omnipotent market maker, who is able to clear the market instantaneously, (a rather common assumption in many financial ABMs [2–4]) and the presence of the exogenous noise. In our previous papers we have speculated that exogenous noise is likely to originate from the order book dynamics. Consequently it seems that including order book dynamics in our financial ABM could potentially resolve these two issues at once. Such approach has an additional perk, it would allow us to consider trading activity time series as well.

Though the empirical order book data has become available at the same time as the empirical high-frequency financial time series data, in the 1980's, it took a longer time for detailed empirical studies of the order book dynamics to be undertaken [13–16]. Interestingly some of the observations were not as universal as the stylized facts discovered in the empirical time series. There are few recent empirical order book studies, which confirm some of the earlier findings, but fail to confirm the other findings [17]. Similarly there is a variety of simple limit order book models (e.g., [18–20]), which are not mutually compatible, but are able to reproduce some of the empirical observations in the order book data. Some of the more recent order book modeling approaches, such as [21–24], are more sophisticated and able to reproduce a variety of empirical observations. The most recent approach by Kanazawa et al. [25,26] combines empirical and theoretical modeling approaches. Namely, Kanazawa et al. have observed the behavior of the high-frequency traders in a highly detailed order book level data set. Based on the observations a microscopic model inspired by the kinetic theory was proposed. Nevertheless most of these models have not considered a detailed reproduction of the stylized facts established for the time series data.

In this paper our goal is to introduce the order book mechanics, similar to the ones introduced in [25,26], into our financial ABM [10,11] thus producing a novel order book model with herd behavior, which would be able to reproduce power law statistical properties of the absolute return and trading activity time series. In Section 2 we will compare the order book modeling approach we will take here against a few similar recently published approaches [27–35]. In Section 3 we will briefly introduce herd behavior model proposed by Kirman [12], which is the basis of our agent-based approach. In Section 4 we will discuss two different financial market models: the previously proposed herd behavior model with instantaneous clearing [10,11] and the order book model with herd behavior. We will show that under certain parameter values both of the models produce mostly identical statistical properties. Further in Section 5 we will compare the order book model with herd behavior against the empirical high-frequency Bitcoin and NYSE data. We will examine the sensitivity of the model to parameter value changes in Section 6. While we will provide concluding remarks and future outlooks in Section 7.

2. Review of the other similar order book modeling approaches

Recently a few approaches similar to the one we will take here were published [27–35]. While these approaches are different in many aspects, we will keep our review brief by highlighting only the most important similarities and differences between these approaches themselves and our approach to be taken in the next sections of this paper. The review is summarized in Table 1.

First of all it is important to note that all of the models under our review are able to reproduce the main stylized facts of absolute return time series. Some of the models are also able to reproduce the features of price time series [27] or specific peculiarities observed in the modeled markets [33,34]. Neither of the considered approaches reports results on the reproducibility of the statistical properties of trading activity. Our goal will be to reproduce PDFs and PSDs of both absolute return and trading activity time series. Note that some of the approaches reviewed here report auto-correlation functions of the return time series, while we prefer to use PSDs. This makes no significant difference as PSD is directly related to the auto-correlation function via the Wiener–Khinchin theorem.

All of the considered models assume that there are two types of traders. One type of traders relies on the internal market information. These traders are usually called chartists. Another type of traders use exogenous information. In most of the works exogenous information is some kind of parametrization of the fundamental price, hence such traders are called chartists. Among the considered works only Cocco et al. [33,34] make a different assumption. Namely, Cocco et al. assume that exogenous information is related to the objective needs of traders, which are unknown and random,

therefore it causes random trading. Our approach is similar to the most of these works, we will assume that the market is populated by the two types of agents: chartists and fundamentalists.

Unlike in these models in our approach agents are flexible, namely they are able to switch their trading strategies. It is quite a puzzle to explain why the models with fixed and flexible agent types are able to reproduce the main stylized facts. Most likely it is because these models rely on the heterogeneity of the agents within each type. Whenever this intra-type heterogeneity breaks down, the agents act similarly on the received information, and thus large fluctuations emerge. In our approach large fluctuations also emerge when most agents become chartists, i.e., when inter-type heterogeneity breaks down. In all cases fluctuations will be the largest under heterogeneity breaking.

The approaches also differ on how the trading strategies are specified. Most realistic specification can be found in [32,35] as the model proposed in these works utilize technical trading strategies which are actually used for trading in the real markets. A bit less practical, yet sound from an economic perspective, strategies can be found in [27,29–31]. These strategies are economically sound because they allow the agents to maximize their expected wealth, though these strategies are not necessarily common among the traders in the real markets. While our approach, and also [28,33,34], is the simplest in this context. Namely, trading strategies in this case are assumed to be highly stylized and do not necessarily guarantee wealth gain for the agents. In some sense it could be said that the agents under this approach have zero intelligence.

We could categorize the approaches to the trading strategies from another point of view. The strategies also significantly differ on how the order volumes are treated. We find three different possible treatments: order volume is selected strategically in [27,35] (it is important to note that neither of these models uses order books), order volume is drawn from random distribution in [29–31,33,34] or it is set to be of unit size [28,32]. In [25,26], on which we will base our order book approach, order volumes are also assumed to be of unit size, this assumption is backed up by empirical analysis, which shows that around 80% of completed transactions fill a unit of volume.

Although the differences in respect to trading strategies can be extremely important when trying to build a model for economic policy making [29–32,35], it does not help to make model more tractable, which is also a rather desirable feature. All of these models, with the exception of [27], are not analytically tractable and can be studied only via numerical simulation. Thus only limited knowledge about their dynamics can be obtained. Though our financial ABM seems to be comparatively lacking in economic plausibility it makes up by being analytically tractable, though one has to rely on Stochastic Calculus for that. As the model in [27] our financial ABM lacks order book clearing mechanism, but in this work we will introduce that in to our financial ABM. As the order book modeling approach we will take in this work is analytically tractable [25,26], we can expect that the combination of the both approaches would also become analytically tractable at some point.

3. Kirman’s herd behavior model

Let us start with discussion about Kirman’s herd behavior model [12], which is the base upon we build financial market models in the following sections. In the seminal paper Kirman shared an observation that social scientists and behavioral biologists observe remarkably similar patterns in rather distinct systems. In the experiments involving ants described by Kirman, entomologists observed the emergence of asymmetry in a symmetric experimental setup: despite having two identical food sources available, majority of the ants in the ant colony preferred to forage from a single food source at a time. Numerous references in Kirman’s paper suggest that humans also seem to prefer the more popular product over the less popular despite both being of a similar quality.

To account for these empirical observations Kirman proposed a simple probabilistic herd behavior model in which the probability for an agent to switch to another state is proportional to the fraction of agents in that state:

$$p(X \rightarrow X + 1) = (N - X) \left(\sigma_1 + h \frac{X}{N} \right) \Delta t, \tag{1}$$

$$p(X \rightarrow X - 1) = X \left[\sigma_2 + h \frac{N - X}{N} \right] \Delta t, \tag{2}$$

where X is a number of agents in the first state, N – a total number of agents, σ_i – an idiosyncratic behavior parameter (encodes preferences for the states), h herd behavior parameter, Δt – arbitrarily small time step. This formulation of the herd behavior model is often referred to as “local” or “extensive”, because the fluctuations of X quickly disappear as N becomes larger [36–39]. In other words X rapidly converges to a certain value and remains almost constant afterwards.

What we described above in the literature is often referred to as the N -dependence problem [36–38]. This problem can be circumvented by assuming that the probability to switch is proportional to a total number of agents X :

$$p(X \rightarrow X + 1) = (N - X) (\sigma_1 + hX) \Delta t, \tag{3}$$

$$p(X \rightarrow X - 1) = X [\sigma_2 + h(N - X)] \Delta t, \tag{4}$$

To contrast the previous formulation of the model, this formulation of the herd behavior model is often referred to as the “non-extensive” or “global” formulation. In this formulation X no longer converges to a fixed value even in the limit of infinite N . This is desirable feature to have in the financial market models as it is well-known that the stylized facts

hold for small and large markets alike [40]. Hence there is a variety of the financial ABMs, which were inspired by the non-extensive formulation of the Kirman's model [9–11,41–44]. There are also papers in the opinion dynamics, which claim that the fluctuating nature of opinion change can be explained by assuming the presence of collective peer-pressure instead of inter-personal communication [37–39,45,46].

Let us take the infinite N limit and introduce an almost continuous state variable $x = \frac{\chi}{N}$. This allows us to rewrite the model driven by Eqs. (3) and (4) as a stochastic differential equation [10,41,42]:

$$dx = h[\varepsilon_1(1-x) - \varepsilon_2x]dt + \sqrt{2hx(1-x)}dW, \quad (5)$$

where $\varepsilon_i = \frac{\sigma_i}{h}$. From the Eq. (5) it is straightforward to conclude that x is Beta distributed, $x \sim \text{Beta}(\varepsilon_1, \varepsilon_2)$. If we would set $\varepsilon_1 = \varepsilon_2$ and $\varepsilon_1 < 1$ then we would observe the same pattern entomologists did as the Beta distribution is multi modal in this case.

One can derive a similar SDE, by taking finitely large N limit under extensive formulation of the model. Yet in this case SDE becomes ordinary differential equation as with larger N the diffusion term becomes negligible and only the drift term remains:

$$\begin{aligned} dx &= h[\varepsilon_1(1-x) - \varepsilon_2x]dt + \sqrt{\frac{2hx(1-x)}{N}}dW \\ &\approx h[\varepsilon_1(1-x) - \varepsilon_2x]dt. \end{aligned} \quad (6)$$

In a special case, $p(X \rightarrow X-1) = 0$, one can also derive the well-known Bass diffusion equation [47]:

$$dx = (1-x)(\sigma_1 + hx)dt. \quad (7)$$

4. Herd behavior model in the context of the financial markets

Kirman's herd behavior model is in a sense a generic model. In the financial market context it would be natural for these states to represent different trading strategies. In the agent-based modeling literature one can find the most common is to consider interaction between agents using fundamentalist and chartist trading strategies [4,6,35]. It is worth to note a couple approaches which consider modeling market sentiments instead [42,48,49]. In our previous works [10,11] we have used both of these approaches to build a highly sophisticated ABM which was able to fit the empirical absolute return PDF and PSD rather well. Here we will start with the fundamentalist-chartist model under instantaneous clearing and later use it to build order book model.

4.1. Herd behavior model with instantaneous clearing

Let us define the trading strategies in a rather stylized manner. One could in general define more sophisticated strategies (as discussed in [35]), but we would like to keep the model compatible with our earlier works. This will allow us to use some of the earlier analytically obtained results which would be impossible while using a more realistic trading strategies.

In [10] we have assumed that excess demand by chartist traders is conditioned on their mood $\xi(t)$:

$$D_c(t) = r_0 N_c(t) \xi(t), \quad (8)$$

where r_0 is the relative impact of chartists' trading activity and $N_c(t)$ is the number of chartists. In contrast fundamentalists' demand is conditioned on their knowledge about the market fundamentals, which is quantified as the fundamental price P_f (which has a physical dimension of generic price units, p.u.):

$$D_f(t) = N_f(t) \ln \frac{P_f}{P(t)} = [N - N_c(t)] \ln \frac{P_f}{P(t)}, \quad (9)$$

where $N_f(t)$ is the number of fundamentalists and $P(t)$ is the current price. Note that here we assume that the fundamental price is fixed, which is not true for the real markets. Nevertheless adding variability to the fundamental price, e.g., by assuming that it follows Brownian motion, would not have a significant impact on the statistical properties of the model.

We have also assumed that a market maker instantaneously clears the market by setting the price to the Walras equilibrium price, which is obtained in the following manner:

$$D_f(t) + D_c(t) = 0 \Rightarrow P_{\text{eq}}(t) = P_f \exp\left(r_0 \cdot \frac{N_c(t)}{N - N_c(t)} \cdot \xi(t)\right). \quad (10)$$

If $\xi(t)$ fluctuates significantly faster than $N_c(t)$, then the absolute return

$$|r(t)| = \left| \ln \frac{P(t)}{P(t-T)} \right| \propto \frac{N_c(t)}{N - N_c(t)} = y. \quad (11)$$

Through out our papers we have referred to y as modulating return as it describes longer-term fluctuations of the absolute return, while $\xi(t)$ dictates the rapid fluctuations and changes in the sign of the return. In some of the earlier works $\xi(t)$ was even modeled as a noise [41].

Previously [10] we have also extended the original herd behavior model by introducing the feedback of the modulating return y on the switching dynamics:

$$p(N_c \rightarrow N_c + 1) = (N - N_c) [\sigma_{fc} + hN_c] \frac{\Delta t}{\tau(N_c)}, \tag{12}$$

$$p(N_c \rightarrow N_c - 1) = N_c [\sigma_{cf} + h(N - N_c)] \frac{\Delta t}{\tau(N_c)}, \tag{13}$$

where

$$\tau(N_c) = \left(\frac{N_c}{N - N_c} \right)^{-\alpha} \equiv y^{-\alpha}. \tag{14}$$

That is, $\tau(N_c)$ adjusts the characteristic time scale of microscopic switching events according to the current global value of the modulating return. Such feedback scenario implements the coupling between returns and trading activity, which is well established empirical fact [50]. Note that in [10] switching dynamics were assumed to correlate with trading activity.

Introduction of the feedback scenario enables us to obtain a more general form of the SDE for y , which has tunable noise multiplicativity exponent:

$$\begin{aligned} dy &= h [\varepsilon_{fc} + (2 - \varepsilon_{cf})y] \frac{1+y}{\tau(y)} dt + \sqrt{\frac{2hy}{\tau(y)}} (1+y) dW \\ &= h [\varepsilon_{fc} + (2 - \varepsilon_{cf})y] (1+y) y^\alpha dt + \sqrt{2hy^{1+\alpha}} (1+y) dW \\ &\approx h (2 - \varepsilon_{cf}) y^{2+\alpha} dt + \sqrt{2hy^{3+\alpha}} dW. \end{aligned} \tag{15}$$

This SDE, assuming $y \gg 1$, belongs to a class of SDEs exhibiting power law statistics described in [51]. Thus it the y time series should exhibit power law statistics [10]:

$$P(y) \sim y^{-\varepsilon_{cf} - \alpha - 1}, \quad S(f) \sim f^{-1 - \frac{\varepsilon_{cf} + \alpha - 2}{1 + \alpha}}. \tag{16}$$

This simple model already reproduces two main stylized facts. In the later papers, e.g., [11], we have extended this model by describing the mood dynamics using the same herd behavior model. Though in order to fit the empirical absolute return PDF and PSD we had to introduce exogenous noise, which we assumed to represent additional randomness arising from the order book dynamics and possibly an exogenous information inflow. In the next section we build the order book model to address this assumption.

4.2. Order book model with herd behavior

Most of the ABMs, which consider statistical properties of the various financial time series, directly or indirectly assume presence of the market maker [4]. While the real financial markets are not cleared by an idealized market maker, most of the contemporary financial markets implement trading by using the order books. Similarly to the market makers order books record and execute orders that the traders submit. The difference is that the orders in the order book are executed only if there is an overlap between the buy (bid) and the ask (sell) sides of the order book or if a market order is submitted. While there is a significant body of literature considering order book modeling [19–26,28], most of these models consider reproducing patterns observed at the order book level and usually neglect stylized facts related to the financial time series. It is worth to note that there are numerous papers in economics, which suggest that the prices in various auctions converge towards Walras equilibrium and that this convergence might be comparatively fast [52–57]. Yet this convergence is not instantaneous and we might observe some interesting effects in the high-frequency financial time series. Here, while building our order book model, we will partly rely on an empirically motivated order book model proposed by Kanazawa et al. [25,26].

As in [25,26] we assume that chartists as high-frequency traders submit unit volume limit orders to the both sides of the order book. The submitted quotes, Q_i^{ask} and Q_i^{bid} , are placed by the i th agent the same distance, S_i , away from the current valuation of the stock, V_i :

$$Q_i^{\text{ask}}(t) = V_i(t) + S_i, \tag{17}$$

$$Q_i^{\text{bid}}(t) = V_i(t) - S_i, \tag{18}$$

$$S_i \sim \text{Gamma}(k, \theta), \tag{19}$$

where k is the shape parameter of the Gamma distribution and θ is the scale parameter of the Gamma distribution. Here physical dimension of θ is p.u. (generic price units) as it is also the dimension of Q_i , V_i and S_i . Further in this paper we

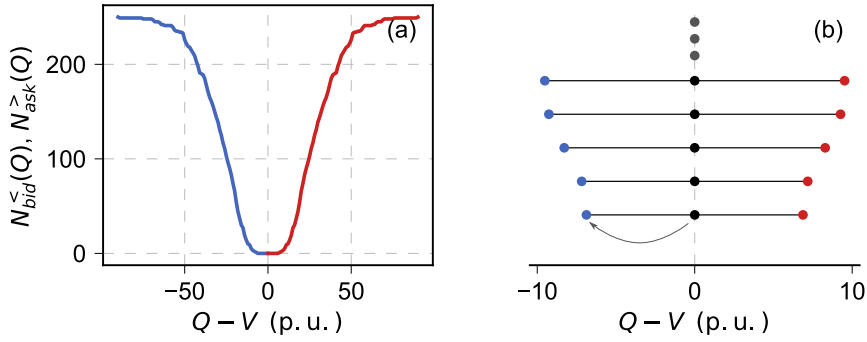


Fig. 1. Full order book profile (a) and a zoom in to 5 best quotes present in the order book (b). Blue curve (a) and circles (b) denote bid (buy) limit orders, red curve (a) and circles (b) denote ask (sell) limit orders, while black circles (b) show V_i for each of the best quotes. In sub figure (a) ask profile is cumulative, giving number of limit orders willing to sell for a given price or larger, while bid profile is inversely cumulative, giving the number of limit orders willing to buy at a given price or smaller. In sub figure (b) the arrow shows where the valuation would move if market ask order would be submitted.

will use the empirically determined values $k = 4$ and $\theta = 15.5$ p.u. (see [25,26]) unless specified otherwise. Note that these values were obtained specifically for USD/JPY exchange rate in Forex and they might take different values for the other markets or exchange rates. Yet we will rely on these values as best available estimate at this point.

Next we will replace the sophisticated trend following mechanism originally present in [25,26] with a simpler market order submission mechanism, which is similar to the one used in the herd behavior model with instantaneous clearing. This simplification leads to another simplifying assumption that valuations are homogeneous $V_i(t) = V(t)$. Hence the valuation changes only after a market order is executed. After a market order is executed, the valuation is set to the value of realized quote. As in the original order book approach, after the valuation is reset chartists update their submitted quotes.

An example of the full order book profile is shown in Fig. 1(a). Sub figure (b) of Fig. 1 provides zoomed in picture of the 5 best quotes, and shows the movement of valuation if market ask order would be submitted. Blue and red circles represent bid and ask quotes respectively, these circles are connected to show that the both quotes are submitted by the same agent, whose valuation is represented by the black circle. Note that under our simplifications the profile is symmetric around quote equal to the current valuation, which is identical for all agents. This would not be the case for the original approach in [25,26].

Now let discuss the replacement of the trend following mechanism of [25,26] with a simpler one used in the herd behavior model with instantaneous clearing. Let the chartists submit unit volume market orders at rate

$$\lambda_{tc}(t) = \frac{\lambda_e}{\tau(N_c(t))} \lambda_{tc} N_c(t), \quad (20)$$

where λ_e is the reference event rate (which has a physical dimension of s^{-1}) and λ_{tc} – the relative market order submission rate by a single chartist agent. The submitted market order is bid order with probability:

$$p_{bid}(t) = \frac{1 + \xi(t)}{2}, \quad (21)$$

and ask market order is submitted otherwise, $p_{ask}(t) = 1 - p_{bid}(t)$. Let us assume that the mood simply flips its sign at rate

$$\lambda_M(t) = \frac{\lambda_e}{\tau(N_c(t))} \lambda_m, \quad (22)$$

where λ_m is relative mood flipping rate. As the sign flip does not change the modulus the mood will take only two possible values, $\xi(t) \in \{-\xi_0, +\xi_0\}$ (here ξ_0 is initial value of the mood).

Fundamentalists are not present in [25,26], so we can keep our earlier assumptions about their behavior. Namely, we assume that fundamentalists are willing to buy stock if there is such ask quote for which $Q_j^{ask}(t) < P_f$, and are willing to sell stock if there is such bid quote for which $Q_j^{bid}(t) > P_f$. They will submit unit volume market orders, if there are suitable quotes in the order book, at rate:

$$\lambda_{tf}(t) = \frac{\lambda_e}{\tau(N_c(t))} \lambda_{tf} [N - N_c(t)] \left| \ln \left(\frac{P(t)}{P_f} \right) \right|, \quad (23)$$

where λ_{tf} is the relative market order submission rate by a single fundamentalist agent.

Note that in [25,26] power law return distribution is obtained only after taking into account the fact that the number of high-frequency traders (chartists) changes over time. If the number of high-frequency traders (chartists) would be fixed,

then returns would be exponentially distributed. This intuition was confirmed empirically by splitting the time series into two hour periods. For each of these periods returns were found to be approximately exponentially distributed, though the different values of rates were found for the different periods. The values of rates were found to be positively correlated with the average number of high-frequency traders (chartists) present during the same periods. Next let us include this variation to the current model by including switching behavior present in the herd behavior model with instantaneous clearing.

It is straightforward to determine event rates for the trading strategy switching: the fundamentalist will switch to the chartist trading strategy at rate,

$$\lambda_{fc}(t) = \frac{\lambda_e}{\tau(N_c(t))} [N - N_c(t)] [\varepsilon_{fc} + N_c(t)], \tag{24}$$

while the chartist will switch to the fundamentalist trading strategy at rate,

$$\lambda_{cf}(t) = \frac{\lambda_e}{\tau(N_c(t))} N_c(t) [\varepsilon_{cf} + \{N - N_c(t)\}]. \tag{25}$$

Note that these transition rates do not include parameter h , which is because it is equivalent to λ_e . As soon as chartist becomes fundamentalist his limit orders are canceled, also if fundamentalist becomes chartist, then he immediately submits his limit orders.

As the number of agents in this model will always be finite, the probability of $\tau(N_c) = 0$ (or alternatively $N_c(t) = N$) and $\tau(N_f) = \infty$ (or alternatively $N_c(t) = 0$) will be non-zero, which would lead to “over-heating” or “freezing” of the strategy switching dynamics. To avoid these edge cases let us redefine the feedback scenario as:

$$\tau^{-1}(N_c(t)) = \lambda_0 + \begin{cases} \text{if } N_c(t) = N & [2N_c(t)]^\alpha \\ \text{else} & \left[\frac{N_c(t)}{N - N_c(t)} \right]^\alpha \end{cases}, \tag{26}$$

where λ_0 is the relative minimum switching rate. In the above we have multiplied N_c by 2 when taking the $N_c = N$ edge case into account, because the previous increase in y , number of chartists increasing from $N_c = N - 2$ to $N_c = N - 1$, is approximately double given large N .

We use the Gillespie algorithm [58,59] to implement the order book model. The main idea behind the Gillespie algorithm is that we can sum all of the event rates to obtain the total event rate:

$$\lambda^T = \lambda_{cf} + \lambda_{fc} + \lambda_M + \lambda_{tF} + \lambda_{tC}, \tag{27}$$

which enables us to generate random inter-event times, which are distributed exponentially:

$$\Delta t_i \sim \text{Exp}(\lambda^T). \tag{28}$$

After each Δt_i one of the possible events happens. The probability for any of the possible events to happen is proportional to its rate:

$$p_{cf} = \frac{\lambda_{cf}}{\lambda^T}, \quad p_{fc} = \frac{\lambda_{fc}}{\lambda^T}, \quad p_M = \frac{\lambda_M}{\lambda^T}, \quad p_{tF} = \frac{\lambda_{tF}}{\lambda^T}, \quad p_{tC} = \frac{\lambda_{tC}}{\lambda^T}. \tag{29}$$

As these probabilities sum to 1, one of the five possible events is bound to happen: either randomly selected chartist switches to fundamentalist trading strategy (with probability p_{cf}) or randomly selected fundamentalist switches to chartist trading strategy (with probability p_{fc}) or the mood flips its sign (with probability p_M) or the randomly selected fundamentalist submits market order (with probability p_{tF}) and randomly selected chartist submits market order (with probability p_{tC}).

The exact algorithm behind this model is summarized as a flowchart in Fig. 2. The code implementing this model is publicly available on <https://github.com/akononovicius/herding-OB-model>. All of the parameters used in this model are summarized in Table 2.

4.3. Comparison between the models

It is possible to approximately estimate equilibrium prices for the order book model. From the discussion in the previous section we know that chartists submit unit volume market orders at rate λ_{tC} , with probability p_{bid} they buy the stock and with probability $1 - p_{bid}$ they sell the stock. Hence their excess demand (on average) is given by:

$$\bar{D}_c = \lambda_{tC} p_{bid} - \lambda_{tC} (1 - p_{bid}) = \frac{\lambda_e}{\tau(N_c(t))} \lambda_{tC} N_c(t) \xi(t). \tag{30}$$

Note that the final result is similar to Eq. (8). Fundamentalists on the other hand submit unit volume market orders at rate λ_{tF} , they submit ask orders if there is such j for which $Q_j^{bid}(t) > P_f$ and bid orders if there is such j for which $Q_j^{ask}(t) < P_f$. Assuming that P_f is not in the spread, the excess demand of fundamentalists (on average) will be given by:

$$\bar{D}_f = \frac{\lambda_e}{\tau(N_c(t))} \lambda_{tF} [N - N_c(t)] \ln \left(\frac{P(t)}{P_f} \right). \tag{31}$$

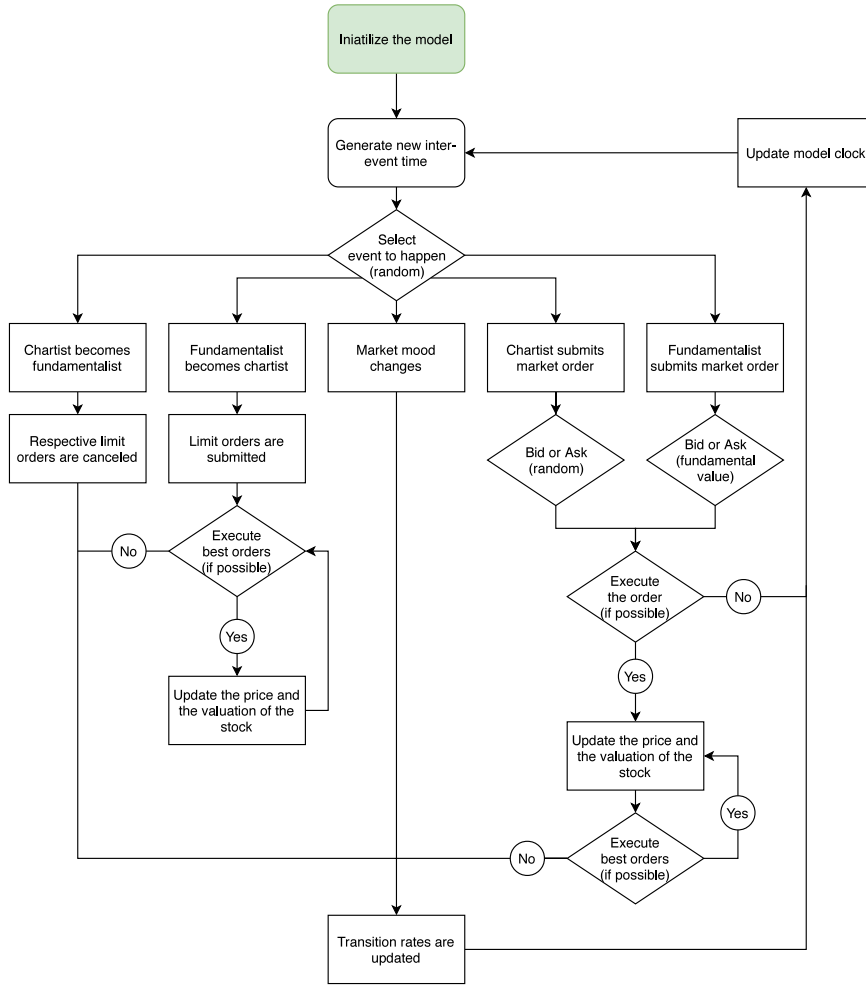


Fig. 2. (Color online) Flowchart illustrating the order book model with herd behavior.

Table 2

List of parameters used in the order book model with herd behavior.

Symbol	Meaning
N	Number of agents
λ_e	Reference event rate (has physical dimension of 1/s)
ϵ_{cf}	Relative idiosyncratic switching rate from chartists to fundamentalists
ϵ_{fc}	Relative idiosyncratic switching rate from fundamentalists to chartists
ξ_0	Absolute value of chartists' mood
λ_m	Relative rate at which mood flips its sign
λ_0	Relative minimum switching rate between chartists and fundamentalists
α	Exponent of the feedback scenario
k	Order book shape parameter
θ	Order book scale parameter (has physical dimension of generic price units, p.u.)
P_f	Fundamental price (has physical dimension of generic price units, p.u.)
λ_{tc}	Relative market order submission rate for chartists
λ_{tf}	Relative market order submission rate for fundamentalists

Note that the final result is similar to Eq. (9). Assuming that order book is non-empty and almost uniformly filled we can obtain the equilibrium price:

$$\bar{D}_c + \bar{D}_f = 0 \Rightarrow P_{eq}(t) = P_f \exp\left(\frac{\lambda_{tc}}{\lambda_{tf}} \cdot \frac{N_c(t)}{N - N_c(t)} \cdot \xi(t)\right). \tag{32}$$

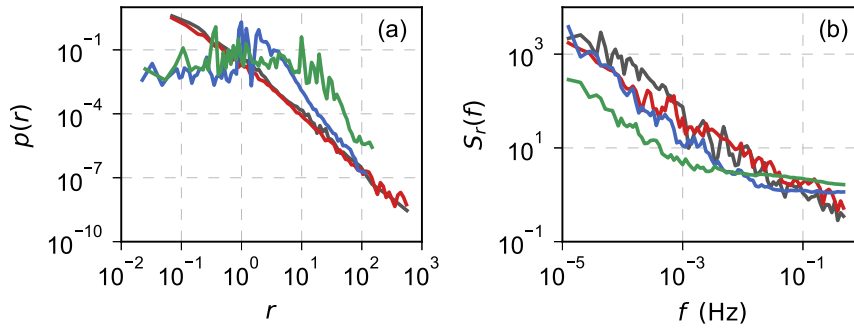


Fig. 3. Comparison between the statistical properties, (a) PDFs and (b) PSDs, of y time series (gray curves) and absolute return time series (red, blue and green curves), black curves show the expected slopes of the statistical properties, as per Eq. (16). For the best comparison all of the time series were normalized to unit standard deviation. The following parameter values were used in numerical simulations: $N = 500$, $\lambda_e = 10^{-7} \text{ s}^{-1}$, $\varepsilon_{ef} = \varepsilon_{fc} = 1$, $\xi_0 = 0.2$, $\lambda_m = 10^3$, $\lambda_0 = 0.1$, $\alpha = 1$, $k = 4$, $\theta = 15.5 \text{ p.u.}$, $P_f = 3 \cdot 10^4 \text{ p.u.}$ (all cases), $\lambda_{tc} = \lambda_{tf} = 3 \cdot 10^4$ (red curves), 300 (blue and gray curves), 3 (green curves).

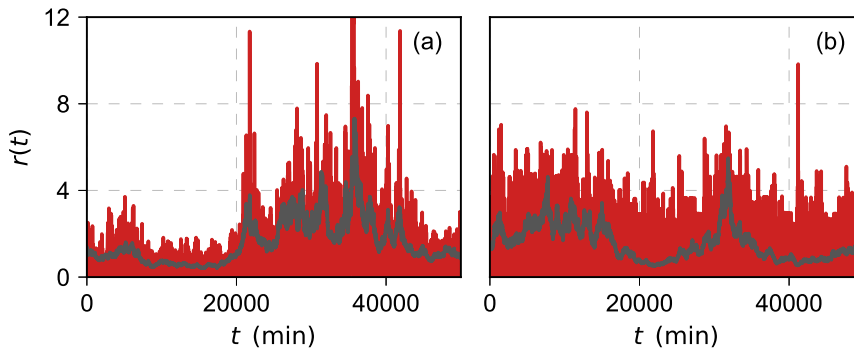


Fig. 4. Sample fragments of the absolute return (red curves) and y (gray curves) time series. Parameter values are identical to the ones used in Fig. 3 except: (a) $\lambda_{tc} = \lambda_{tf} = 3 \cdot 10^4$ (red curve in Fig. 3), (b) $\lambda_{tc} = \lambda_{tf} = 3$ (green curve in Fig. 3). The correlation coefficients between the samples are $\rho = 0.67$ (a) and 0.03 (b).

As the expression for the equilibrium price has the same form as in Eq. (10), we can expect that y and return will have similar statistical properties. The main condition we have to ensure for the similarity to be observable is that enough trades happen between N_c changes, so that the equilibrium price could be reached. This means that λ_{tc} and λ_{tf} have to be rather large. As you can see in Fig. 3 the agreement between the statistical properties of y and absolute return improves as λ_{tc} and λ_{tf} grow larger. In Fig. 4 we have shown sample y and absolute return time series, which can be seen to correlate in sub figure (a) and be almost uncorrelated in sub figure (b).

Correlation between y and absolute return time series, see sample series in Fig. 4, also is stronger with larger λ_{tc} and λ_{tf} . For the series shown in Fig. 4(a) correlation is strong $\rho \approx 0.7$ (with $\lambda_{tc} = \lambda_{tf} = 3 \cdot 10^4$), while in Fig. 4(b) correlation is negligible $\rho \approx 0$ (with $\lambda_{tc} = \lambda_{tf} = 3$).

Similar intuition can be obtained from Fig. 5. As you can see in (a) and (c), for large λ_{tc} and λ_{tf} (parameters the same as for the red curve in Fig. 3) the price tends to catch up with the changes in the equilibrium price. Though the following is far from being perfect as can be seen by zooming in on the series, (c). The correlation between the price and the equilibrium price time series is mild. While for small λ_{tc} and λ_{tf} , (b) and (d), (parameters the same as for the green curve in Fig. 3) it is evident that the price does not manage to catch up with the changes in the equilibrium price. As expected, there is almost no correlation between the time series.

Based on these results we would like to argue that the fracture in the PSD of the high-frequency absolute return time series happens due to order book dynamics. Namely, the absolute return PSD in the high frequency range is less sloped, because the markets are unable to discover the equilibrium price that fast. It seems that it could take a day or two (as the fracture is usually between 10^{-5} and 10^{-4} Hz) for the markets to discover the new equilibrium price.

5. Comparison against the empirical data

Before making a comparison against the empirical data let us state that neither order book model presented in [25,26] (which forms a basis of our order book approach) nor our financial ABM [10,11] (which forms a basis of our agent-based approach) is able to reproduce all of the empirical statistical properties considered further in this Section. Our earlier

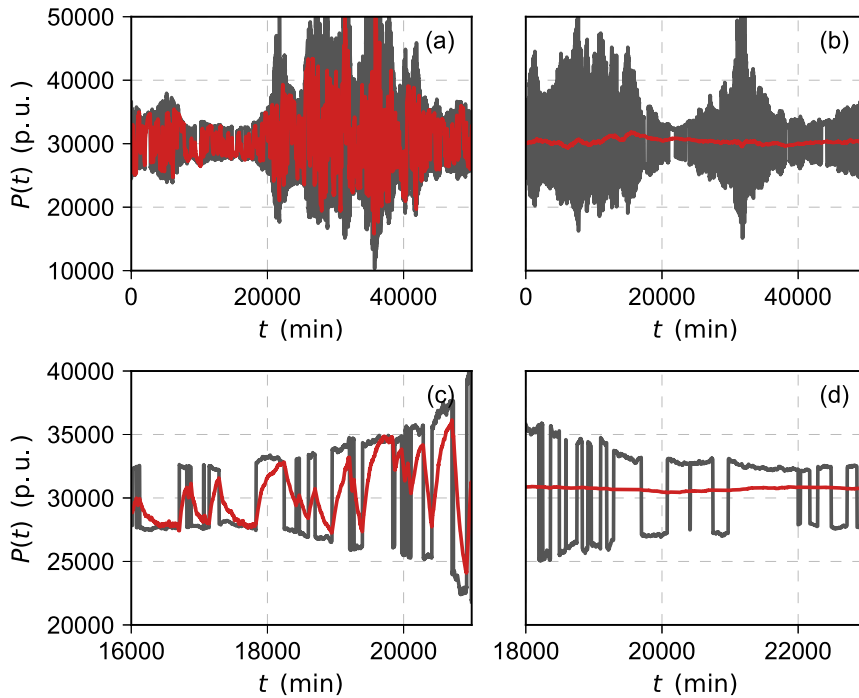


Fig. 5. Sample fragments of the price (red curves) and the equilibrium price (gray curves) time series. Parameter values are identical to the ones used in Fig. 3 except: (a) and (c) $\lambda_{tc} = \lambda_{tf} = 3 \cdot 10^4$ (red curve in Fig. 3), (b) and (d) $\lambda_{tc} = \lambda_{tf} = 3$ (green curve in Fig. 3). The correlation coefficients between the samples are $\rho = 0.54$ (a) and 0.02 (b).

Table 3

List of the considered Bitcoin time series.

Exchange	Exchange pair	Period available	Period used	Trades
bitfinex	BTC/USD	2013-03-31/2016-12-22	From 2013-05-01	10^7 (99.4%)
bitflyer	BTC/JPY	2017-07-04/2018-07-04	Whole	$3 \cdot 10^7$
bitstamp	BTC/USD	2011-09-13/2018-07-04	Except 2016-06-23	$2.6 \cdot 10^7$ (99.9%)
btcbbox	BTC/JPY	2014-04-09/2018-07-04	Whole	$8.9 \cdot 10^6$
btce	BTC/USD	2011-08-14/2017-07-25	Whole	$3.3 \cdot 10^7$
btcn	BTC/CNY	2011-06-13/2017-09-30	From 2013-04-01	$1.1 \cdot 10^8$ (99.9%)
btcoind	BTC/IDR	2014-02-09/2018-07-04	Whole	$8 \cdot 10^6$
btctrade	BTC/CNY	2013-05-19/2017-09-30	Whole	$2 \cdot 10^7$
coinbase	BTC/EUR	2015-04-23/2018-07-04	Whole	$1.5 \cdot 10^7$
coinbase	BTC/USD	2014-12-01/2018-07-04	From 2017-05-01	$3 \cdot 10^7$ (67.5%)
coincheck	BTC/JPY	2014-10-31/2018-07-04	Except 2017-08-07	10^8 (99.9%)
kraken	BTC/EUR	2014-01-08/2018-07-04	Whole	$2.1 \cdot 10^7$
kraken	BTC/USD	2014-01-07/2018-07-04	Whole	10^7
okcoin	BTC/CNY	2013-06-12/2015-04-05	Whole	10^8

financial ABM [10,11] was shown to reproduce statistical properties of absolute return reasonably well, yet there was no way to consider trading activity in that model. While [25,26] has focused on reproduction of order book dynamics only briefly touching upon statistical properties of absolute return.

In this paper we use publicly available tick by tick trading data from 12 different Bitcoin exchanges. We have downloaded the data from bitcoincharts.com website on July 5, 2018. List of the considered Bitcoin time series is given in Table 3. Note that Coinbase and Kraken exchanges appear twice in the table, because they contribute more than one exchange pair. These time series were selected, because their data files were among top 5% of the largest. Fisco's BTC/JPY, Zai's BTC/JPY and Zyado's BTC/EUR were also among top 5% of the largest, but these time series were excluded, because their statistical properties were too different from the rest of the time series. Note that for the same reason we have truncated some of the Bitcoin time series which remained under our consideration.

For each of the considered Bitcoin time series we have produced one minute absolute return (normalized to the standard deviation) and trading activity (defined as trades per time interval and normalized to the mean) time series. For each of the produced one minute time series we have calculated PDF and PSD. The obtained statistical properties were averaged to produce average profile for each of the statistical properties. To select the model parameters we have

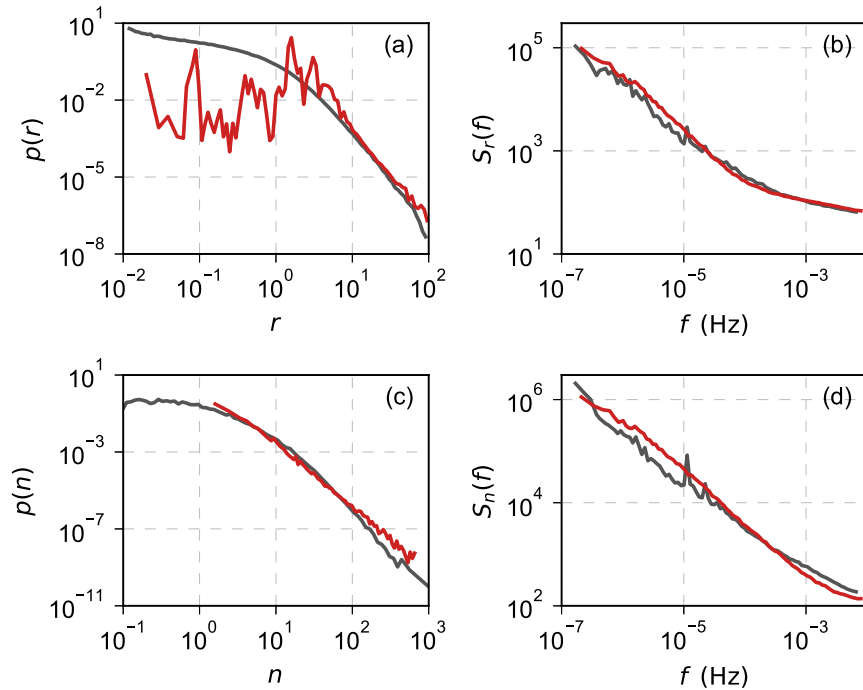


Fig. 6. Comparison between the empirical Bitcoin statistical properties (gray curves) and statistical properties generated by the model (red curves): (a) one minute absolute return PDF, (b) one minute absolute return PSD, (c) trading activity per one minute PDF, (d) trading activity per one minute PSD. The following parameter set was used in numerical simulations: $N = 500$, $\lambda_e = 10^{-7} \text{ s}^{-1}$, $\varepsilon_{fc} = 5$, $\varepsilon_{cf} = 2$, $\xi_0 = 0.2$, $\lambda_m = 10$, $\lambda_{tc} = 25$, $\lambda_{tf} = 75$, $\lambda_0 = 0.4$, $\alpha = 2$, $k = 4$, $\theta = 15.5 \text{ p.u.}$, $P_f = 3 \cdot 10^4 \text{ p.u.}$

used simulated annealing technique with a goal to reproduce these averaged statistical properties. As you can see in Fig. 6 the obtained agreement is rather good.

Earlier works, such as the ones found in [60,61], has already carried out detailed analysis of the Bitcoin time series and established that the stylized facts for the Bitcoin are somewhat different from the stylized facts established for the stocks. Namely, [61] has reported that the Bitcoin returns exhibit heavier tails than ordinary stock returns. Similar finding was also reported in [60]. Yet these works disagree on whether the tail index changes over time. In [61] it was reported that while the tail index has slightly increased, but not enough statistical evidence was found to formally claim that the tail index is increasing. In [60], on the other hand, a significant change in the tail index is reported. Furthermore other statistical properties of absolute return, such as slope of the auto-correlation function, Hurst exponent and multi-scaling properties, seem to approach values observed for the ordinary stocks. This is interpreted as a sign that Bitcoin market is becoming “mature” market. In this paper we did not carry out a rigorous empirical analysis, at least as rigorous as in [60,61]. We have just checked whether PDFs and PSDs of absolute return and trading activity change over time and found that the change, if present, is negligible for the exchange rates that have remained under our consideration. While the return tail index seems to be similar to the one obtained for NYSE stocks under our consideration ($\lambda \approx 3.7$; analysis of NYSE stocks are discussed in the next paragraph). A thorough analysis of the Bitcoin time series or a meta analysis of the methods used in [60,61] would be due, but this topic is out of the scope of this paper.

We have also considered the statistical properties of 26 tickers from NYSE. The considered tickers include: ABT, ADM, BA, BMY, C, CVX, DOW, FON, FNM, GE, GM, HD, IBM, JNJ, JPM, KO, LLY, MMM, MO, MOT, MRK, SLE, PFE, T, WMT and XOM. All their time frames are from January, 2005 to March, 2007. As with the Bitcoin time series, we have obtained averaged statistical properties for NYSE data set. Using simulated annealing technique we have obtained another best fit parameter set for our model. To obtain a better fit we had to divide absolute return time series generated by the model by factor of 3. This indicates that the model still lacks something, though it seems to reproduce correct behavior for the tail of the distribution. As we can see in Fig. 7 after this correction the agreement between the model and the data appears to be rather good. The obtained parameter set is similar to the one obtained for the Bitcoin case. Though there are some differences. The mood swings seem to be larger in NYSE case (the respective ξ_0 is larger), while Bitcoin trading would seem to be more random (which is somewhat consistent with assumptions made in [33,34]). On the other hand chartists seem to submit less order in NYSE case (λ_{tc} is smaller), which would indicate prices in NYSE a more impacted by the market fundamentals. Base trading activity seems to be higher, λ_0 is larger, for NYSE. The last two differences could potentially indicate that NYSE is a more mature market than Bitcoin exchanges, as is also noted based on empirical observations in [60].

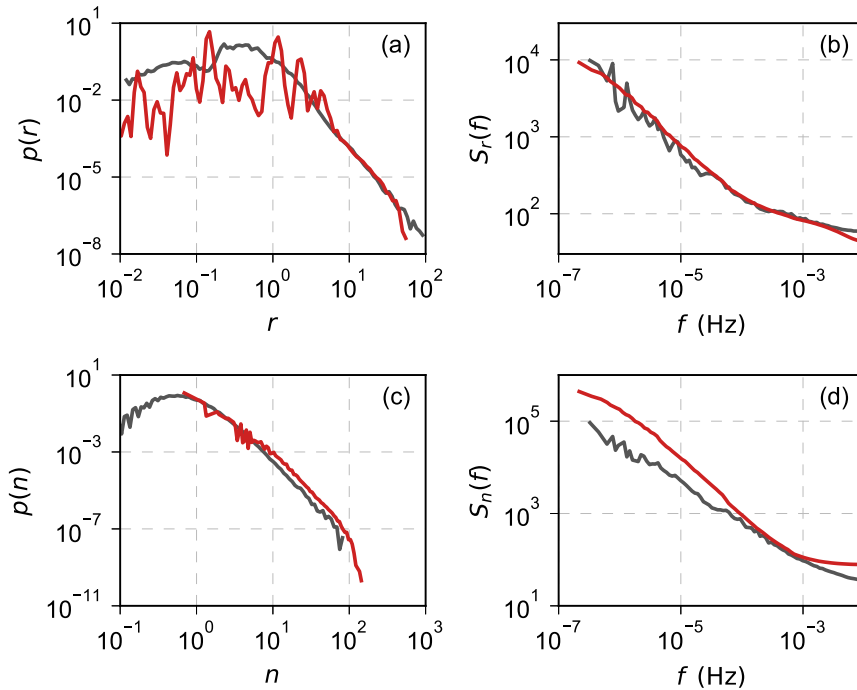


Fig. 7. Comparison between the empirical NYSE stocks' statistical properties (gray curves) and statistical properties generated by the model (red curves). Parameter values are identical to the ones used in Fig. 6 except: $\xi_0 = 1$, $\lambda_{tc} = 2$, $\lambda_0 = 1.5$.

6. Impact of the model parameters

In this section we check to see how changing the model's parameter values impact the statistical properties of absolute return and trading activity generated by the model. In all figures in this section we will show three curves. Usually one will be generated with a larger parameter value than used to produce Fig. 6 (blue curve), one smaller (green curve) and one identical (red curve).

Models built on the non-extensive formulation of the herd behavior model are known to avoid the N -dependence problem [10,11,36–39,41,42], but as we can see in Fig. 8 this model has some kind of N -dependence. Yet the fluctuations do not disappear with larger N , it seems that the model starts to exhibit even fatter tails. This is most likely occurs due to the implemented mood mechanism: the more agents and the more chartist agents, the more mood is reflected in the time series.

We can partly eliminate this dependence by recalling that while y dynamics are not influenced by N , but the number of trades per time window does depend on N . By requiring that $N\lambda_{tc} = \text{const}$ and $N\lambda_{cf} = \text{const}$ we eliminate this dependence. And as we can see in Fig. 9 then changing N does not influence the statistical properties of absolute return. Though N retains the impact on the trading activity.

Changing λ_e parameter value seems to have a similar impact as changing h in the original herd behavior model (see Fig. 10). Namely the PSD of the absolute return shifts to the right as we increase λ_e . Though due to the absolute return formula comparing log-prices at two different points in time, the PDF of the absolute return might also be impacted: it seem that the PDF might obtain heavier tails as λ_e becomes larger. Interestingly changing λ_e does not seem to have any qualitative effect on the statistical properties of the trading activity. Most likely increasing λ_e simply increase the mean of trades per time window without changing anything else.

Parameter ε_{fc} does not have significant impact on the statistical properties of the model with instantaneous clearing, Eq. (16), but it seems that it is able to impact the statistical properties of the absolute return in the order book model (see Fig. 11). As ε_{fc} increases the tails of the PDF becomes heavier and the PSD becomes flatter. The statistical properties of the trading activity do not seem to change qualitatively, the tail of the PDF remains the same nor does the steepness of the PSD change. Most likely larger ε_{fc} simply increases mean trading activity.

Parameter ε_{cf} seems to have the opposite effect on the statistical properties of absolute return (see Fig. 12). As ε_{cf} increases the tail of the PDF become lighter, while the slope of the PSD becomes steeper. The impact on the statistical properties of the trading activity seems to be both quantitative, the mean number of trades per time window decreases as ε_{cf} increases, and qualitative, the tail of the PDF becomes lighter and the slope of the PSD becomes steeper. These effects are most likely caused by the dynamics reflected by the model with instantaneous clearing as such dependence is predicted by Eq. (16).

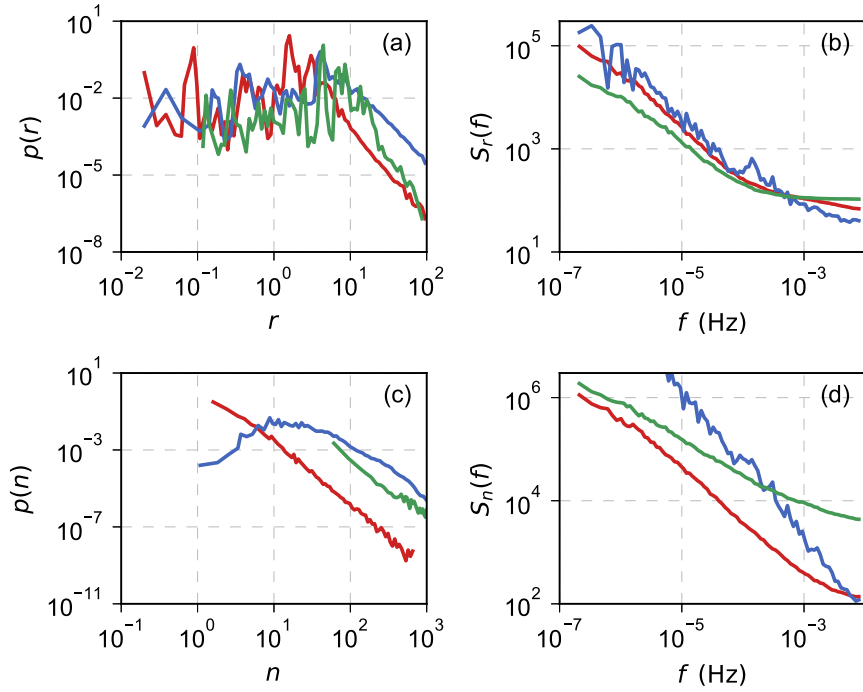


Fig. 8. Influence of N parameter on the model's statistical properties. Parameter values are identical to the ones used in Fig. 6 except: $N = 5000$ (blue curve), $N = 50$ (green curve).

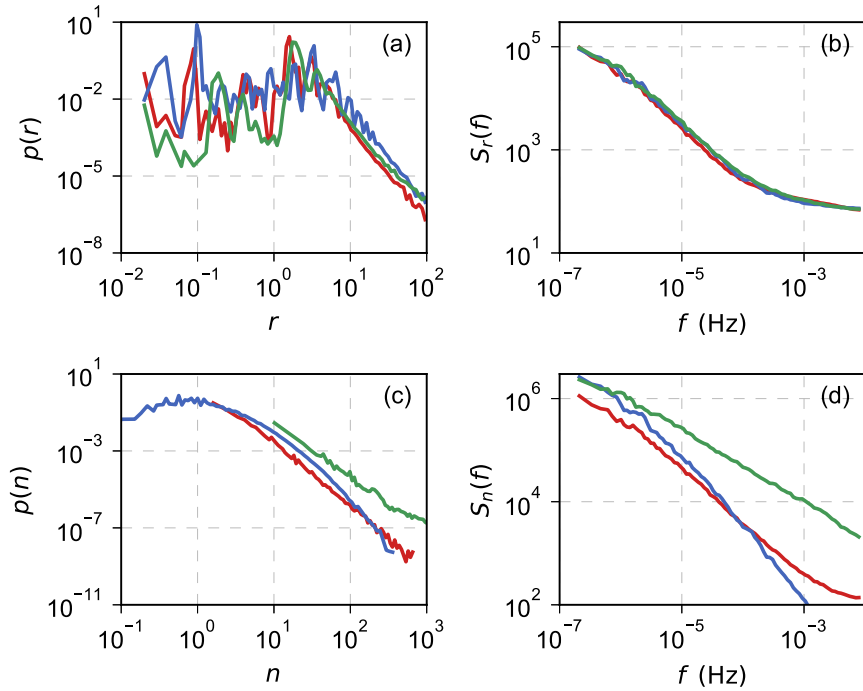


Fig. 9. Influence of N parameter on the model's statistical properties, when $N\lambda_{tc} = \text{const}$ and $N\lambda_{tf} = \text{const}$. Parameter values are identical to the ones used in Fig. 6 except: $N = 5000, \lambda_{tc} = 2.5, \lambda_{tf} = 7.5$ (blue curve), $N = 50, \lambda_{tc} = 250, \lambda_{tf} = 750$ (green curve).

The mood dynamics, ξ_0 and λ_{mc} , does not seem to have a significant impact on the statistical properties of both absolute return and trading activity (see Figs. 13 and 14). Though small effect of ξ_0 on the absolute return PDF and PSD are visible. Larger ξ_0 makes the tail of the PDF less fat and the PSD flatter. Changing λ_{mc} would have a larger if the base value of ξ_0

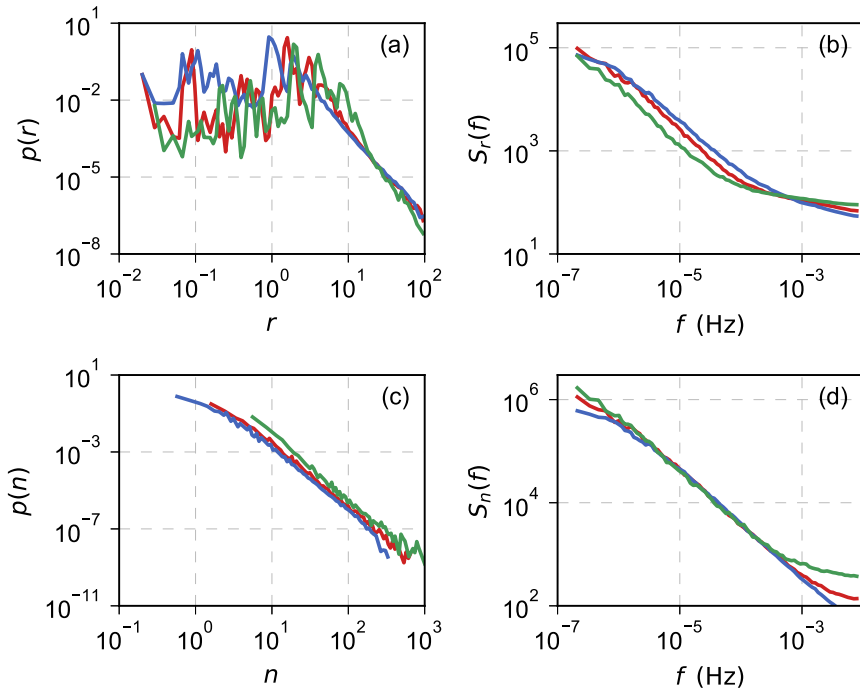


Fig. 10. Influence of λ_e parameter on the model's statistical properties. Parameter values are identical to the ones used in Fig. 6 except: $\lambda_e = 3 \cdot 10^{-7} \text{ s}^{-1}$ (blue curve), $\lambda_e = 3 \cdot 10^{-8} \text{ s}^{-1}$ (green curve).

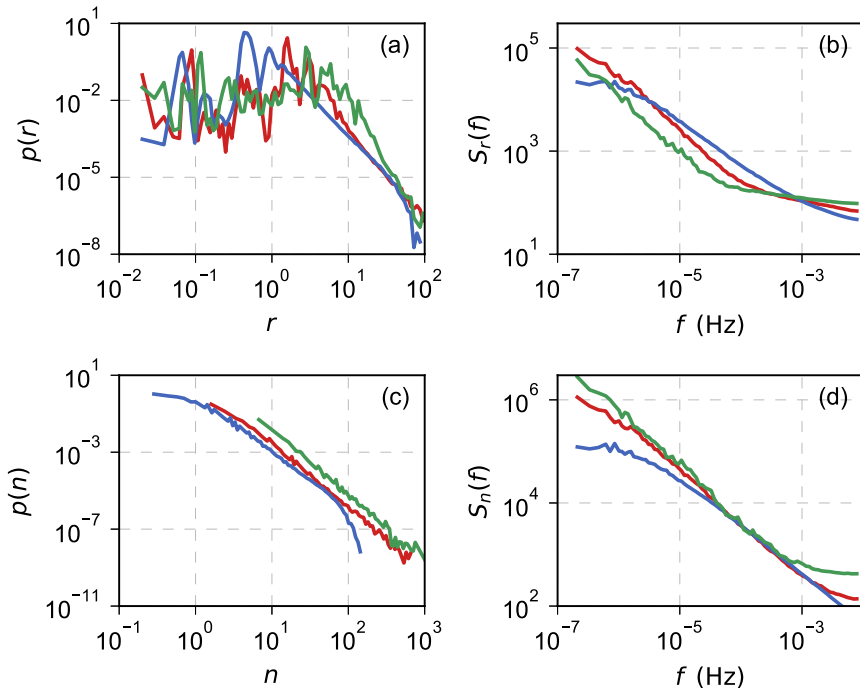


Fig. 11. Influence of ϵ_{fc} parameter on the model's statistical properties. Parameter values are identical to the ones used in Fig. 6 except: $\epsilon_{fc} = 15$ (blue curve), $\epsilon_{fc} = 0.7$ (green curve).

was larger. Increasing λ_{mc} would have similar effect as decreasing ξ_0 as with larger λ_m the effective mood (the average trend) would be smaller than the true value of ξ_0 .

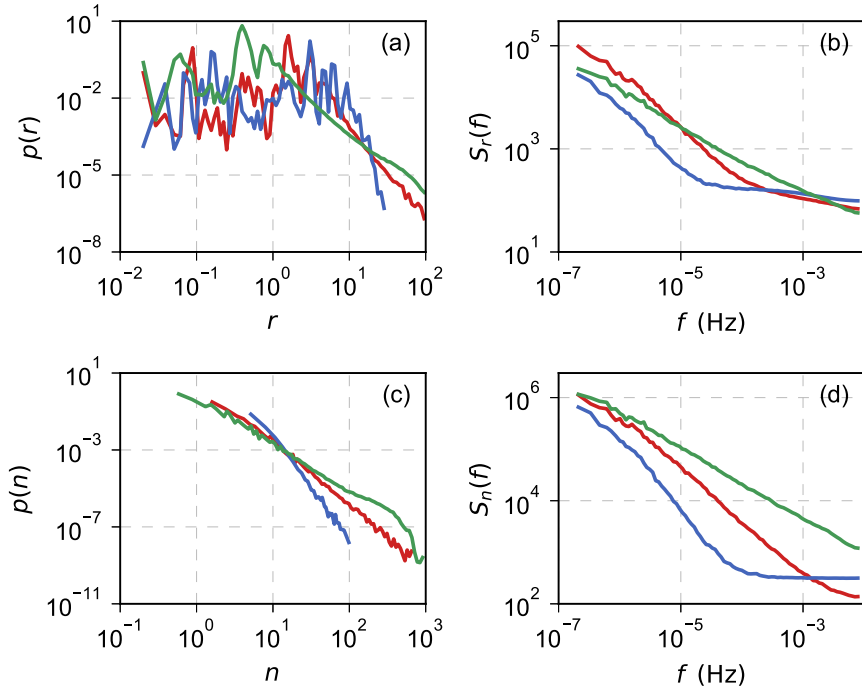


Fig. 12. Influence of ε_{cf} parameter on the model's statistical properties. Parameter values are identical to the ones used in Fig. 6 except: $\varepsilon_{cf} = 6$ (blue curve), $\varepsilon_{cf} = 0.7$ (green curve).

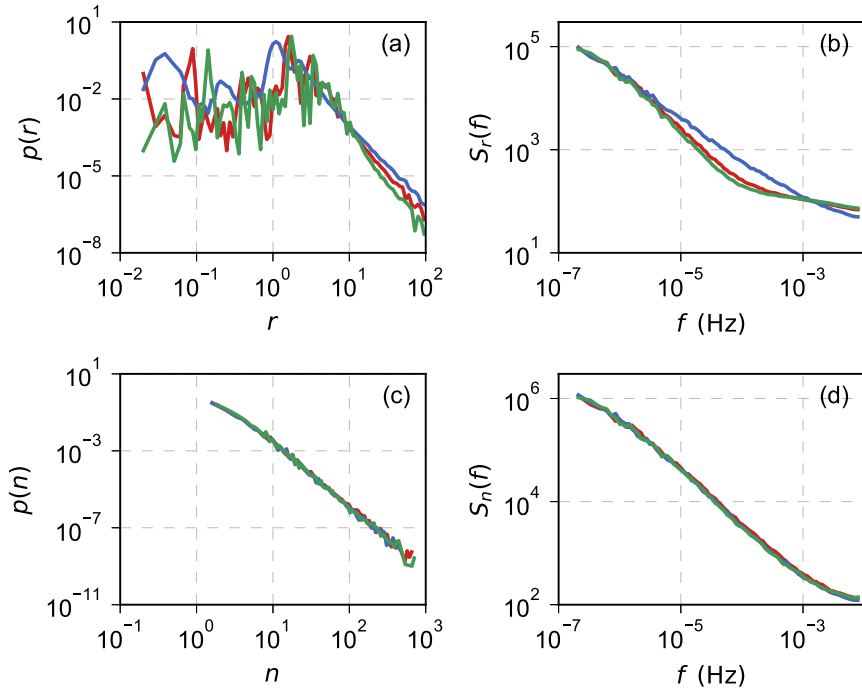


Fig. 13. Influence of ξ_0 parameter on the model's statistical properties. Parameter values are identical to the ones used in Fig. 6 except: $\xi_0 = 0.6$ (blue curve), $\xi_0 = 0.1$ (green curve).

As we have seen in the previous section larger λ_{tc} and λ_{tf} values force the realized prices to more closely follow the equilibrium prices. While looking at the statistical properties of the model we see that λ_{tf} does not seem to have a noticeable effect (see Fig. 16). This is because the fundamentalists activate only if the current price deviates far from the

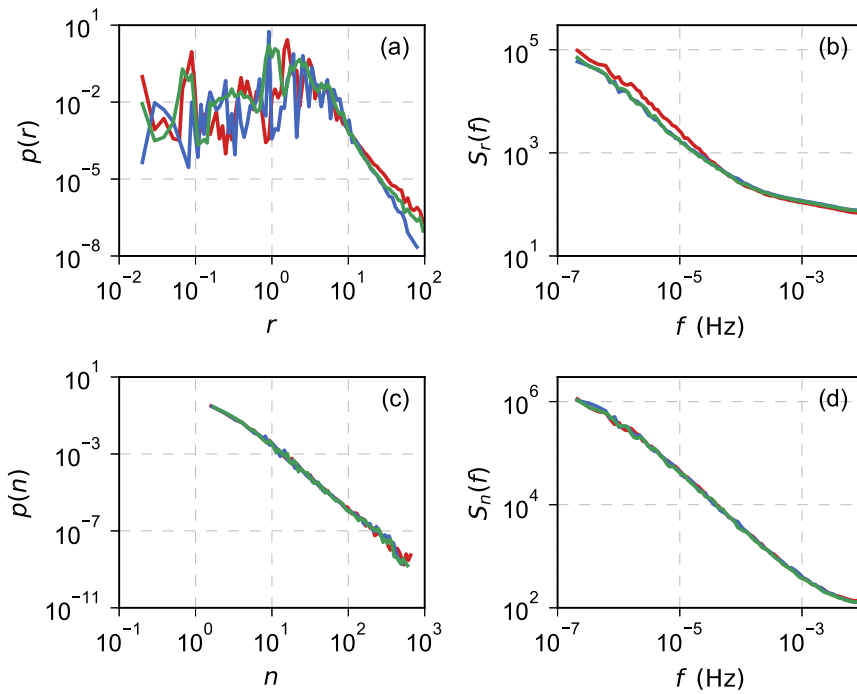


Fig. 14. Influence of λ_m parameter on the model's statistical properties. Parameter values are identical to the ones used in Fig. 6 except: $\lambda_m = 300$ (blue curve), $\lambda_m = 0.3$ (green curve).

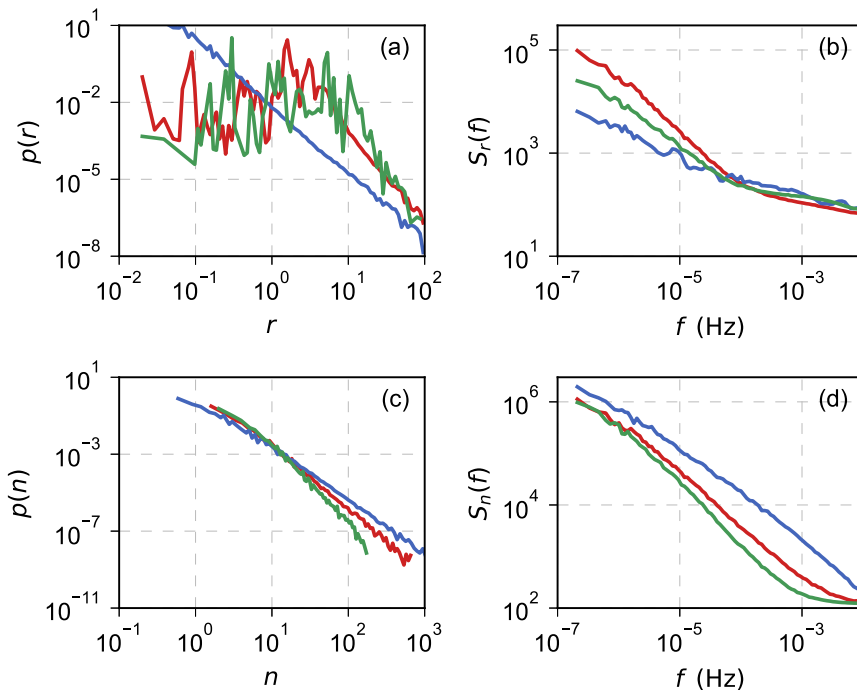


Fig. 15. Influence of λ_{tc} parameter on the model's statistical properties. Parameter values are identical to the ones used in Fig. 6 except: $\lambda_{tc} = 250$ (blue curve), $\lambda_{tc} = 2.5$ (green curve).

fundamental price and their trades rapidly push the price back to the fundamental price. On the other hand λ_{tc} seems to have a significant effect (see Fig. 15): larger values lead to the fatter tails of the PDFs, while the PSDs flatten.

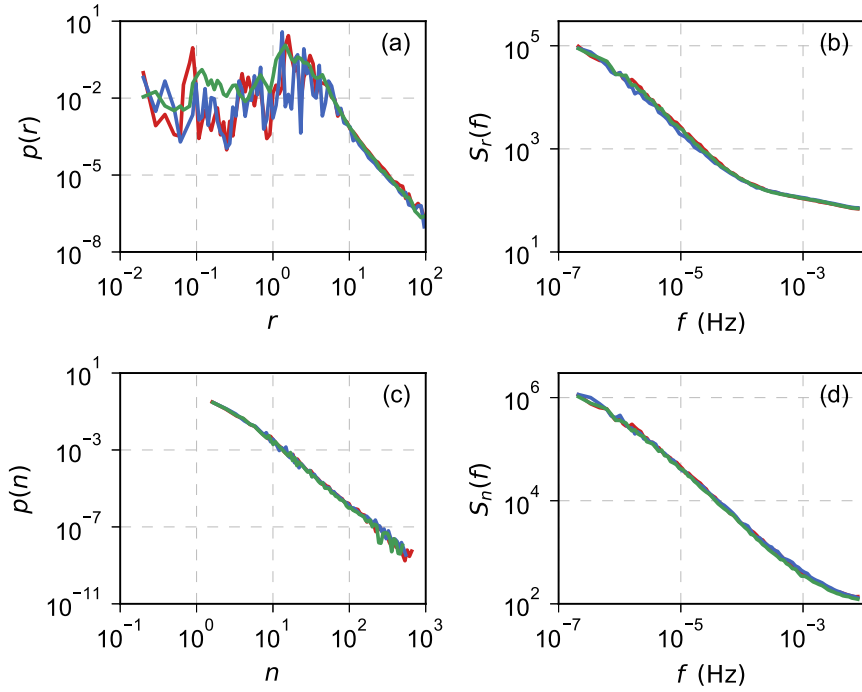


Fig. 16. Influence of λ_{tf} parameter on the model's statistical properties. Parameter values are identical to the ones used in Fig. 6 except: $\lambda_{tf} = 750$ (blue curve), $\lambda_{tc} = 7.5$ (green curve).

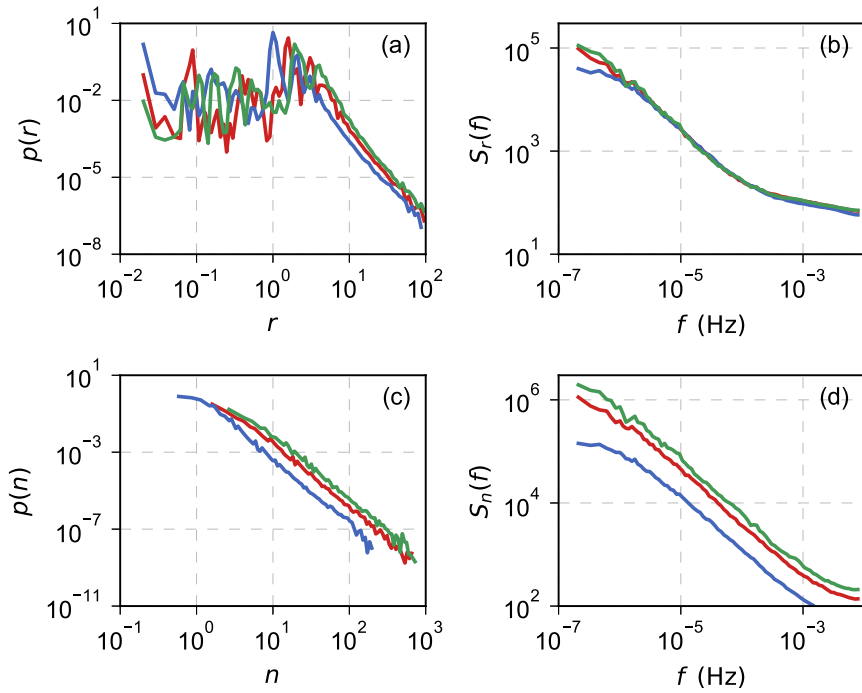


Fig. 17. Influence of λ_0 parameter on the model's statistical properties. Parameter values are identical to the ones used in Fig. 6 except: $\lambda_0 = 4$ (blue curve), $\lambda_0 = 0.04$ (green curve).

As we can see in Fig. 17 changing λ_0 values does not have a significant effect besides increasing the overall level of trading activity.

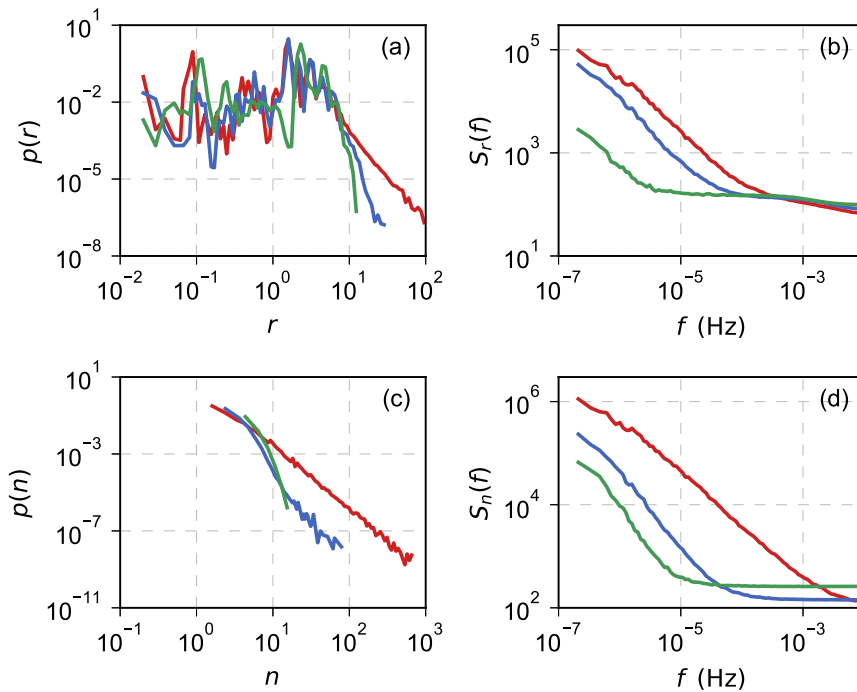


Fig. 18. Influence of α parameter on the model's statistical properties. Parameter values are identical to the ones used in Fig. 6 except: $\alpha = 1$ (blue curve), $\alpha = 0$ (green curve).

Changing the power of the feedback scenario α seems to have an adverse effect (see Fig. 18): with smaller values the tails of the PDFs become lighter and the slopes of the PSDs become steeper.

Interestingly, as can be seen in Figs. 19 and 20, the parameters influencing the overall shape of the order book itself, k and θ , do not seem to have any effect on the statistical properties obtained from the normalized time series. This result could be attributed to the simplifying assumption about the homogeneity of the valuation. If valuation would be allowed to be heterogeneous as in [25,26], the order book shape parameters would likely have a more profound effect on the observed statistical properties.

7. Conclusions

Here we have proposed an order book model with herd behavior, which is able to reproduce the main stylized facts of the financial markets. The order book model with herd behavior was built upon empirical insights by Kanazawa et al. [25,26], who have studied a very detailed records of the order book level events, and our previously proposed theoretical ABM [10,11], which is known to successfully reproduce the statistical properties of the high-frequency absolute return. Incorporating order book dynamics improves upon our previous work in numerous ways. First of all, we were able to scrap two not very realistic, but still common in the literature, assumptions: we no longer need to introduce an efficient market maker to define the market price (introduced in [10]), we also no longer need to introduce the exogenous noise as was done in [11]. Another key improvement is that now we are able to consider statistical properties of the trading activity alongside the statistical properties of absolute return.

Using simulated annealing, optimizing the root mean squared error of the worst match, we were able to calibrate the model parameters to match the Bitcoin's statistical properties on one minute timescale. Calibrating the model to match the statistical properties observed in NYSE (one minute timescale) was not as successful, which indicates that the model still lacks something. We believe that introducing heterogeneity into chartist and fundamentalist valuation of the stock might be the key, but this would further complicate the model introducing additional parameters. Another possibility would be to complicate the modeling of the chartist mood swings. Finally, the model structure itself suggests that some of the parameter values could be gleaned from the order book level data. Currently we are gathering publicly available Bitcoin order book level data in hopes to use the collected data to improve calibration of the model.

To limit the complexity of the model and to allow for analytical tractability we have made few highly simplifying assumptions. Most restrictive of them are homogeneous valuation, unit order volume and mood dynamics assumptions. As relaxing these assumptions could enrich the dynamics of the proposed order book model with herding behavior, we consider doing so in the future works.

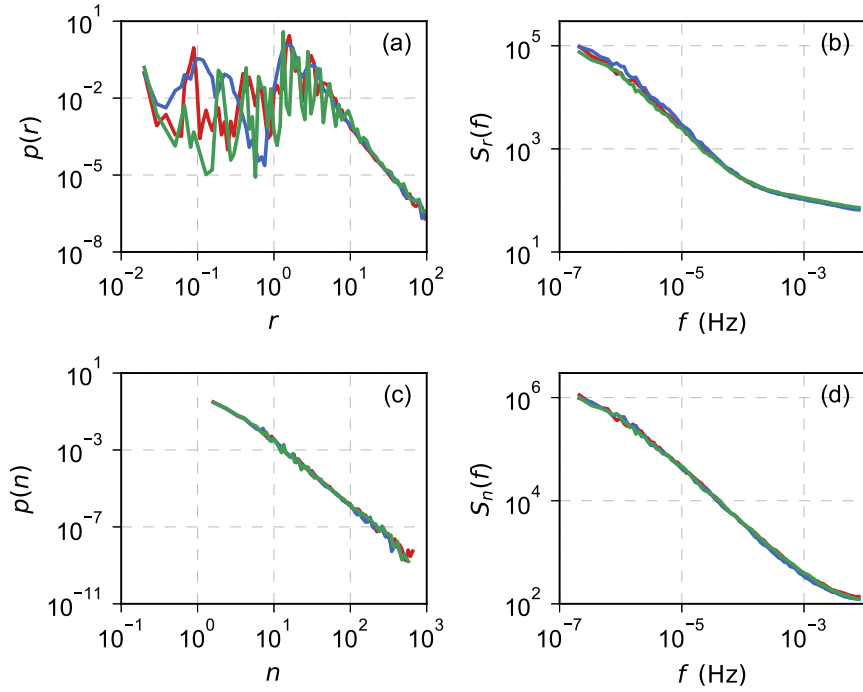


Fig. 19. Influence of k parameter on the model's statistical properties. Parameter values are identical to the ones used in Fig. 6 except: $k = 16$ (blue curve), $k = 1$ (green curve).

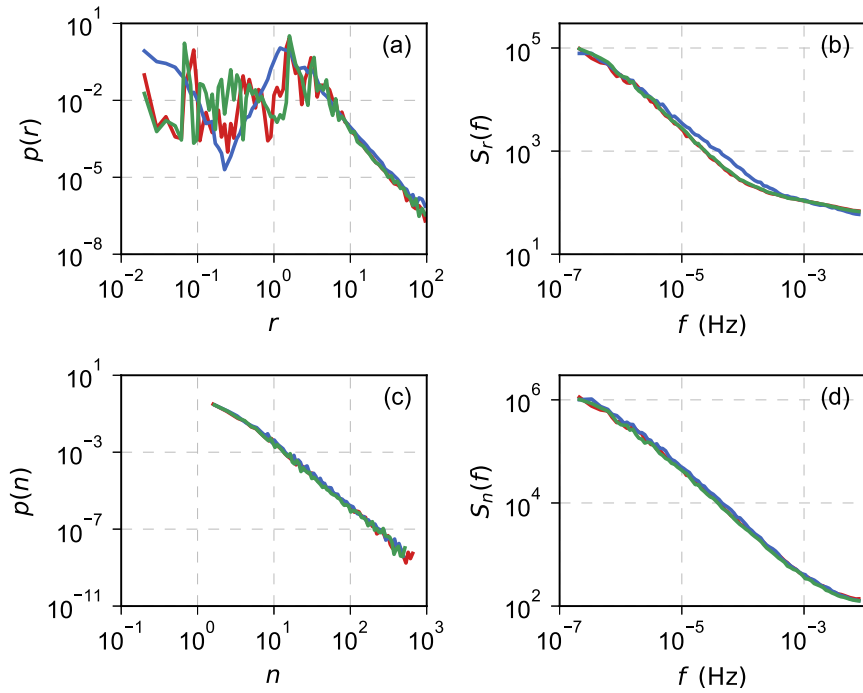


Fig. 20. Influence of θ parameter on the model's statistical properties. Parameter values are identical to the ones used in Fig. 6 except: $\theta = 100$ p.u. (blue curve), $\theta = 2.5$ p.u. (green curve).

A relevant future goal could be improvement economic soundness of the trading strategies and switching behavior of the agents. This would make model more easily comparable with the other recent approaches [27,29–32,35] as well

as open up the possibility to provide a deeper insight for the economic policy makers. Similar transition was already undertaken by Biondo (starting from [28] and arriving to [29–31]).

We could also extend our approach further by considering its dynamics analytically. This should be possible, because one of the underlying models can be alternatively described using stochastic differential equations, while the order book part of the model can also be approached analytically using Bogoliubov–Born–Green–Kirkwood–Yvon hierarchy. Though fully integrating the both parts could prove to be a challenging task. There could be a couple of possible approaches: a superstatistical or a coupled SDE approach as used in [62,63] or a coarse-grained approximation of the model as discussed in [64].

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