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Stochastic dynamics of N correlated binary variables and non-extensive statistical mechanics

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ABSTRACT

The non-extensive statistical mechanics has been applied to describe a variety of complex systems with inherent correlations and feedback loops. Here we present a dynamical model based on previously proposed static model exhibiting in the thermodynamic limit the extensivity of the Tsallis entropy with $q < 1$ as well as a q-Gaussian distribution. The dynamical model consists of a one-dimensional ring of particles characterized by correlated binary random variables, which are allowed to flip according to a simple random walk rule. The proposed dynamical model provides an insight how a mesoscopic dynamics characterized by the non-extensive statistical mechanics could emerge from a microscopic description of the system.

1. Introduction

Assumption that the velocities of the colliding particles are uncorrelated and independent of the position of the particles is a key point in the molecular chaos hypothesis. The hypothesis itself is a cornerstone of the classical (extensive) statistical mechanics. In a more general form the lack of correlations assumption is included into the functional form of the Boltzmann–Gibbs entropy:

$$S_{\mathrm{BG}} = -\int P(x)\ln P(x)\,dx\,. \tag{1}$$

This assumption works rather well for many classical dynamical systems, in which large number of particles and their interactions help to mask the existing correlations. E.g., after the collision of gas particles the velocities of those particles are no longer uncorrelated, but due to large number of them and, thus, large number of the collisions between them, the correlation quickly becomes forgotten. Yet there are systems in which long-range interactions, long-range memory or non-ergodicity are present. To understand these, sometimes referred to as complex, systems and their apparently anomalous properties a generalization of classical statistical mechanics proposed by Tsallis [1] is used. This generalized framework has found its applications to extremely different systems studied by both "hard" (e.g., mathematics, physics, chemistry) and "soft" (e.g., economics) sciences [2–9].

The non-extensive statistical mechanics framework is constructed starting from the generalization of the Boltzmann–Gibbs entropy [1]

$$S_q = \frac{1-\int [P(x)]^q\,dx}{q-1}\,, \tag{2}$$

where $P(x)$ is a probability density function of finding the system in the state characterized by the parameter x, the parameter q describes the non-extensiveness of the system. The original Boltzmann–Gibbs entropy, Eq. (1), can be obtained from Eq. (2) in the limit $q \to 1$ [1,10]. More generalized entropies and distribution functions are introduced in Refs. [11,12]. In the classical statistical mechanics the Gaussian distribution plays an important role. Similarly, in the non-extensive statistical mechanics the q-Gaussian distribution

$$P_q(x) = C\exp_q(-A_q x^2) \tag{3}$$

becomes important [1]. Note that in Eq. (3) q-exponential function is present, this function is defined as follows

$$\exp_q(x) \equiv [1+(1-q)x]_+^{\frac{1}{1-q}}\,, \tag{4}$$

where the notation $[x]_+$ means $[x]_+ = x$ if $x > 0$, and $[x]_+ = 0$ otherwise. Recently, in Ref. [13] it has been shown that q-Gaussian distributions emerge in the limit of large number of realizations in a generalized binomial distribution representing a sequence of correlated trials.

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To illustrate non-extensive statistical mechanics several statistical models which in the $N \to \infty$ limit provide q-Gaussian attractors have been constructed [14–17]. The first attempt to create such models has been undertaken in Ref. [18]. However, it has been shown that the distributions do not approach a q-Gaussian form when the number of particles N in the model increases [19]. Two models that provide q-Gaussian distributions have been introduced in [14], the second model does so by construction. More detailed analysis of the models in [14] has been presented in [15] and the generalization to higher dimensions has been proposed in [16]. However, the standard Boltzmann–Gibbs entropy remains extensive for the models from Ref. [14]. This situation has been improved by the model presented in Ref. [17]. This model is based on a system composed of N distinguishable particles arranged in a chain, each of the particles in the chain is characterized by a binary random variable (this can be, for example, particles with the spin $\frac{1}{2}$). In the aforementioned model it is assumed that spins next to each other are almost always aligned in the same direction. The only exceptions are assumed to be d cases in which the neighboring spins are antialigned. In this model the number of states grows as a power-law, with the corresponding index q being $q_{\text{stat}} = 1 - 1/d$. In addition, the distribution of the total spin of the system in the limit of $N \to \infty$ tends to a q-Gaussian with $q_{\text{dist}} \neq q_{\text{stat}}$. All models in [14,17,18], except the last model of [14] are for $q \leq 1$.

The distributions of the non-extensive statistical mechanics can be obtained from mesoscopic description of the systems in the form of probabilistic dynamics [1]. Such probabilistic description can be provided by nonlinear Fokker–Planck equations and corresponding nonlinear stochastic differential equations (SDEs) [20,21], SDEs with additive and multiplicative noises [22,23] or with multiplicative noise only [24], and with fluctuating friction forces [25]. Nonlinear SDEs generating distributions of non-extensive statistical mechanics together with $1/f$ noise have been proposed in Ref. [26]. Scaling law that follows from nonlinear Fokker–Planck equations has been recently experimentally confirmed in granular media [27].

The goal of this paper is to extend the model presented in Ref. [17] by introducing stochastic temporal dynamics. Conserving the total number of domains d, we allow spins near domain boundaries to flip. This effectively introduces a random walk of domain boundaries into the previous static model. In the limit of large number of spins we derive the Fokker–Planck equation and a corresponding SDE describing the temporal dynamics of the total spin of the system. The proposed dynamical model provides an insight how a mesoscopic dynamics characterized by the non-extensive statistical mechanics could emerge from a microscopic description of the system.

The Letter is organized as follows. To show how stochastic temporal dynamics can be introduced into a model describing the equilibrium, in Section 2 we introduce stochastic dynamics into a simple model consisting of uncorrelated binary random variables. In Section 3 we use a similar stochastic temporal dynamics for the model with correlated binary random variables and leading to extensive generalized entropy with $q < 1$. In Section 4 we demonstrate that presence of macroscopic fluctuations in this model can lead to $1/f$ noise, whereas in Section 5 we consider properties of a part of larger system. We summarize the paper in Section 6.

2. Stochastic dynamics in the model with uncorrelated binary variables

As in Ref. [17], we start with the analysis of the dynamical model without any correlations between the particles or their spins. Let us say that we have N particles which have spin projections on certain axis equal to either $-\frac{1}{2}$ or $+\frac{1}{2}$. The microscopic state of such system is fully described by a set of spin projections $\{s_1, s_2, \ldots, s_N\}$. This system may be treated, in the statistical mechanics sense, using the microcanonical ensemble in which each of the microscopic states is assigned the same probability. Here we obtain the same stationary probability density function (PDF) from the temporal dynamics perspective.

We are interested in an observable macroscopic quantity, the total spin of the system

$$M = \sum_{i=1}^{N} s_i \,. \tag{5}$$

Alternatively we can express the total spin as

$$M = \frac{1}{2}(N_+ - N_-) \,, \tag{6}$$

where N_+ is the number of spins with projection $+\frac{1}{2}$ and N_- is the number of spins with projection $-\frac{1}{2}$. A sum of N_+ and N_-, by definition, should give the total number of spins N,

$$N_+ + N_- = N \,. \tag{7}$$

From Eqs. (6) and (7) we obtain relation between total spin and number of particles having certain spin projections:

$$N_+ = \frac{1}{2}N + M \,, \qquad N_- = \frac{1}{2}N - M \,. \tag{8}$$

To introduce temporal dynamics we assume that during the short time interval Δt each spin can flip with the probability $\frac{1}{2}\gamma\Delta t$. If Δt is short enough we can assume that only one spin flip takes place during the time interval Δt. Since each spin can flip independently, the transition probabilities per unit time are

$$p(M \to M+1) \equiv p^+(M) = \frac{1}{2}\gamma N_- \,, \tag{9}$$

$$p(M \to M-1) \equiv p^-(M) = \frac{1}{2}\gamma N_+ \,. \tag{10}$$

The transition probabilities $p^+(M)$ and $p^-(M)$ define a one-step stochastic process and imply the following Master equation for the probability $P_M(M,t)$ to find the value M of the total spin at time t [28]:

$$\begin{aligned} \frac{\partial}{\partial t}P_M(M,t) = {} & p^+(M-1)P_M(M-1,t) \\ & + p^-(M+1)P_M(M+1,t) \\ & - (p^+(M) + p^-(M))P_M(M,t) \,. \end{aligned} \tag{11}$$

For large enough N we can represent the dynamics by a continuous variable $x = 2M/N$. Using the standard methods from Ref. [28] one can derive the Fokker–Planck equation from the Master equation (11) assuming that N is large and neglecting the terms of the order of $1/N^2$. The resulting Fokker–Planck equation is

$$\begin{aligned} \frac{\partial}{\partial t}P_x(x,t) = {} & -\frac{\partial}{\partial x}[\pi^+(x) - \pi^-(x)]P_x(x,t) \\ & + \frac{\partial^2}{\partial x^2}\frac{1}{N}[\pi^+(x) + \pi^-(x)]P_x(x,t) \,, \end{aligned} \tag{12}$$

where $P_x(x,t) = (N/2)P_M(Nx/2,t)$ is the PDF of the stochastic variable x and

$$\pi^{\pm}(x) \equiv \frac{2}{N}p^{\pm}\left(\frac{Nx}{2}\right) . \tag{13}$$

The SDE corresponding to the Fokker–Planck equation (12) is [29]

$$\frac{dx}{dt} = [\pi^+(x) - \pi^-(x)] + \sqrt{\frac{2}{N}[\pi^+(x) + \pi^-(x)]}\xi(t) \,. \tag{14}$$

Here $\xi(t)$ is a white noise with autocorrelation $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$. When the diffusion coefficient depends on x, SDE (14) should be understood in Itô convention.

For the transition probabilities (9) and (10), the Fokker–Planck equation (12) becomes

$$\frac{\partial}{\partial t}P_x(x,t) = \gamma\frac{\partial}{\partial x}xP_x(x,t) + \frac{\gamma}{N}\frac{\partial^2}{\partial x^2}P_x(x,t) \tag{15}$$

and Eq. (14) takes the form

$$\frac{dx}{dt} = -\gamma x + \sqrt{\frac{2\gamma}{N}}\xi(t)\,. \tag{16}$$

As one can see from Eq. (16), the fluctuations of the macroscopic quantity M decrease with the increasing number of spins as $1/\sqrt{N}$. From Eq. (16) follows that the average value of x obeys the ordinary differential equation

$$\frac{d}{dt}\langle x\rangle = -\gamma\langle x\rangle\,. \tag{17}$$

Thus the deviations from the equilibrium average $\langle x\rangle_0 = 0$ decay exponentially: $\langle x(t)\rangle = \langle x(0)\rangle e^{-\gamma t}$. The steady-state PDF of the stochastic variable x obtained from the Fokker–Planck equation (15) is Gaussian,

$$P_0(x) = \sqrt{\frac{N}{2\pi}}\exp\left(-\frac{N}{2}x^2\right)\,. \tag{18}$$

Gaussian distribution, coinciding with Eq. (18), is also obtained assuming equal probabilities for each microscopic configuration.

3. Stochastic dynamics in the model with correlated binary variables

In this section we investigate the statistical model consisting of N correlated binary random variables, similar to the model of Ref. [17]. As in the previous section one can consider particles having spins $\frac{1}{2}$. In order to avoid special treatment of the ends we will consider the spins situated on a ring instead of one-dimensional chain as in Ref. [17]. We assume that the spins are correlated, meaning that the two adjacent spins are aligned, except for the d cases when the two adjacent spins are antialigned. Note that for a ring of spins only even d is possible. Thus the ring consists of d domains with aligned spins and has d domain boundaries where the spin flips occur. Similarly as in Ref. [17] one can show that the number of allowed microscopic configurations in the model grows with the number N of spins as N^d and the equilibrium distribution of the total spin M in the limit $N\to\infty$ is a q-Gaussian with $q_{\text{dist}} = 1 - 2/(d-2)$.

We introduce stochastic temporal dynamics into the model similarly as we have done in Section 2: we assume that during the short time interval Δt spins can flip with the probability $\frac{1}{2}\gamma\Delta t$. However, in order to conserve d, only the spins next to the domain boundaries are allowed to flip. Furthermore, in order to conserve d, spins belonging to domains containing single spin should not be allowed to flip. If such spin would be allowed to flip, the domains could disappear and d would not be conserved. Alternatively, these spin-flip rules may be seen as introducing a random walk of the domain boundaries. Namely, during the short time interval Δt each domain boundary can move to the left or to the right with equal probabilities $\frac{1}{2}\gamma\Delta t$ unless the move results in two boundaries in the same position. As in Section 2 we will obtain the Fokker–Planck equation describing how the distribution of the total spin changes in time.

If the time interval Δt is short enough, we may assume that only one spin flip takes place at the time. Since spins situated next to the boundaries may flip independently, in the case with no one-spin domains the transition probabilities per unit time are given by

$$p(M\to M+1) = p(M\to M-1) = \frac{1}{2}\gamma d\,. \tag{19}$$

Evidently the presence of one-spin domains would make spin flip less probable. Given only the number of spins N and the total spin M the number of one-spin domains is not exactly known, therefore the actual transition probabilities corresponding to the same total spin M will change over time. For the analytical description of the model we will use averaged in time transition probabilities corresponding to given M. If $K_+(M)$ is the average number of one-spin domains with spin projection $+\frac{1}{2}$ and $K_-(M)$ is the average number of one-spin domains with spin projection $-\frac{1}{2}$, then the average transition probabilities per unit time are

$$\langle p(M\to M+1)\rangle = \frac{1}{2}\gamma(d-2K_-)\,. \tag{20}$$

$$\langle p(M\to M-1)\rangle = \frac{1}{2}\gamma(d-2K_+)\,. \tag{21}$$

We will assume that during the temporal evolution the probabilities of different microscopic configurations are almost equal. Then the average number of one-spin domains can be expressed as

$$K_\pm = \sum_{i=0}^{\frac{d}{2}} i\frac{W_i\left(N_\pm,\frac{d}{2}\right)}{W_{\text{div}}\left(N_\pm,\frac{d}{2}\right)}\,, \tag{22}$$

where $W_{\text{div}}(N_\pm, d/2)$ is the number of possible divisions of $N_\pm$ spins into $d/2$ domains and $W_i(N_\pm, d/2)$ is the number of possible divisions of $N_\pm$ spins into $d/2$ domains such that there are i one-spin domains. The number N_+ of spins with projection $+\frac{1}{2}$ and the number N_- of spins with projection $-\frac{1}{2}$ may be obtained from the known N and M using Eq. (8). The spins with projection $+\frac{1}{2}$ as well as the spins with projection $-\frac{1}{2}$ are divided into $d/2$ domains. Thus the number of possible divisions of spins with the same projection into domains is

$$W_{\text{div}}\left(N_\pm,\frac{d}{2}\right) = \binom{N_\pm-1}{\frac{d}{2}-1}\,. \tag{23}$$

Here $\binom{m}{n}$ is the binomial coefficient. The number of possible configurations with i one-spin domains can be expressed as

$$W_i\left(N_\pm,\frac{d}{2}\right) = \binom{\frac{d}{2}}{i}W_0\left(N_\pm-i,\frac{d}{2}-i\right)\,. \tag{24}$$

Here the binomial coefficient $\binom{d/2}{i}$ gives the number of ways to choose i one-spin domains from $d/2$ domains, while $W_0(N_\pm - i, d/2-i)$ is the number of ways to distribute the remaining $N_\pm - i$ spins into the remaining $d/2-i$ domains containing more than one spin. We obtain the expression for the number $W_0(n,k)$ of possible divisions of n spins into k domains with no one-spin domains as follows. First we put aside k spins. Next we distribute the remaining $n-k$ spins into k domains with at least one spin each. Afterwards we add one spin to each of the k domains using the spins put aside in the first step. Evidently there is only one way to complete the last step, thus the number $W_0(n,k)$ is determined by the number of ways to divide $n-k$ spins into k domains:

$$W_0(n,k) = \binom{n-1-k}{k-1}\,. \tag{25}$$

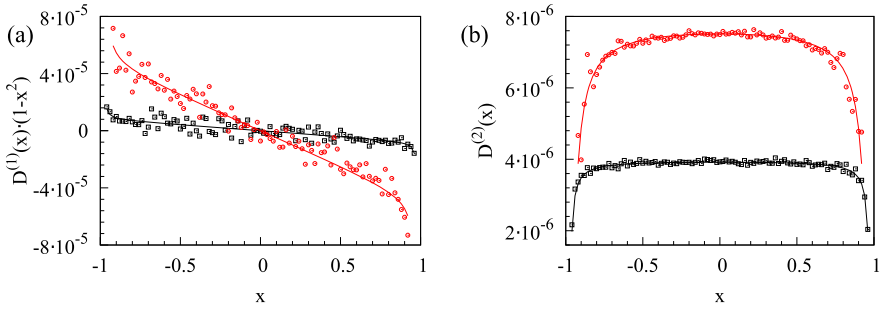

Fig. 1. Kramers–Moyal coefficients (a) $D^{(1)}(x)$ and (b) $D^{(2)}(x)$ (symbols) for two different numbers of spin flips estimated using numerical time series. In order to get a curve closer to a straight line the coefficient $D^{(1)}(x)$ is multiplied by $1-x^2$. Solid lines show corresponding analytic expressions (34) and (35). The number of spins is $N=100$, the number of spin flips is $d=4$ (black squares) and $d=8$ (red circles). Other parameters are $\gamma=0.005$, $\Delta t=1$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Using Eqs. (22)–(25) we obtain the average number of one-spin domains

$$K_{\pm}=\frac{d}{2}\frac{\frac{d}{2}-1}{N_{\pm}-1}. \tag{26}$$

Inserting Eq. (26) into Eqs. (20) and (21) we get the average transition probabilities per unit time

$$\langle p(M\to M+1)\rangle=\frac{\gamma d}{2}\frac{N_{-}-\frac{d}{2}}{N_{-}-1}, \tag{27}$$

$$\langle p(M\to M-1)\rangle=\frac{\gamma d}{2}\frac{N_{+}-\frac{d}{2}}{N_{+}-1}. \tag{28}$$

Using Eqs. (27) and (28) in Eq. (12) and keeping the terms of the order of $1/N^2$ we obtain the Fokker–Planck equation

$$\frac{\partial}{\partial t}P_x(x,t)=\frac{4\gamma d}{N^2}\left(\frac{d}{2}-1\right)\frac{\partial}{\partial x}\frac{x}{1-x^2}P_x(x,t)+\frac{2\gamma d}{N^2}\frac{\partial^2}{\partial x^2}P_x(x,t). \tag{29}$$

The corresponding SDE, according to Eq. (14), is

$$\frac{dx}{dt}=-\frac{4\gamma d}{N^2}\left(\frac{d}{2}-1\right)\frac{x}{1-x^2}+\frac{2}{N}\sqrt{\gamma d}\xi(t). \tag{30}$$

The dependence on the number of spins N in SDE (30) can be removed by introducing the scaled time $t_s=\frac{4\gamma d}{N^2}t$:

$$\frac{dx}{dt_s}=-\left(\frac{d}{2}-1\right)\frac{x}{1-x^2}+\xi(t_s). \tag{31}$$

From Eq. (30) follows that the evolution of the total spin slows down with increasing number of spins N as $1/N^2$. This is in contrast to the case of uncorrelated spins, Eq. (16), where the relaxation towards equilibrium does not depend on N. For the model of correlated spins the fluctuations of the total spin remain macroscopic even in the limit $N\to\infty$, the size of fluctuations of x does not depend on the number of spins N. This is different than in the model of uncorrelated spins where the size of fluctuations decreases as $1/\sqrt{N}$. Steady-state PDF of the stochastic variable x obtained from the Fokker–Planck equation (29) is

$$P_0(x)\propto(1-x^2)^{\frac{d}{2}-1}. \tag{32}$$

This steady-state PDF has a q-Gaussian form with $q<1$ and coincides with the distribution of the total spin obtained assuming equal probabilities for each microscopic configuration. The Fokker–Planck equation (29) is a particular case of known Fokker–Planck equations giving q-Gaussian distributions and satisfies the condition given by Eq. (11) of Ref. [30].

3.1. Testing the Markovian approximation of the dynamical model with correlated binary variables

To check the assumptions made in deriving the Fokker–Planck equation (29) we numerically simulate the temporal evolution of the ring of spins and obtain the time series describing the time dependence of the total spin M. Using those time series we estimate Kramers–Moyal coefficients and compare them to the analytical predictions. In addition we compare time-dependent PDFs of the total spin to the PDFs obtained using Markovian approximation with the transition probabilities per unit time (27), (28).

In numerical simulation we have chosen a random initial configuration with $N=100$ spins and $d=4$ or $d=8$ spin flips, a small time step Δt such that $\gamma\Delta t\ll 1$ and generated time series with $3\cdot 10^7$ points. Using the numerical time series we estimate Kramers–Moyal coefficients [31]

$$D^{(n)}(X)=\frac{1}{n!\Delta t}\left\langle[x(t+\Delta t)-x(t)]^n\right\rangle\big|_{x(t)=X}, \tag{33}$$

where $x=2M/N$. The numerically obtained Kramers–Moyal coefficients $D^{(1)}(x)$ and $D^{(2)}(x)$ are shown as symbols in Fig. 1. We have found the coefficients $D^{(3)}$ and $D^{(4)}$ to be 4 to 5 degrees of magnitude smaller than $D^{(1)}$ and $D^{(2)}$. Thus, according to the Pawula theorem the dynamics of the complete model may be approximated by an Fokker–Planck equation. The continuous curves in Fig. 1 show the analytical expressions for the Kramers–Moyal coefficients

$$D^{(1)}(x)=-\frac{4\gamma d}{N^2}\frac{\left(\frac{d}{2}-1\right)x}{\left(1-\frac{2}{N}\right)^2-x^2}, \tag{34}$$

$$D^{(2)}(x)=\frac{2\gamma d}{N^2}\frac{\left(1-\frac{d}{N}\right)\left(1-\frac{2}{N}\right)-x^2}{\left(1-\frac{2}{N}\right)^2-x^2}, \tag{35}$$

that take into account the effects of the finite number of the spins. To derive Eqs. (34), (35) we have inserted Eqs. (27) and (28) into Eq. (12) without making any further approximations. We see in Fig. 1 a good agreement of the numerically estimated Kramers–Moyal coefficients with the analytic predictions. In the limit of $N\to\infty$ the curves in Fig. 1 should be straight lines. However, as Eqs. (34), (35) and the numerical simulations show, there are deviations from the straight lines for the most extreme values of the total spin. The range of x values where the deviations are significant decreases with increasing N.

Total spin of the system does not determine the transition probabilities completely, because these probabilities depend also on the number of one-spin domains. To check the impact of this missing information on the temporal evolution of the system we have

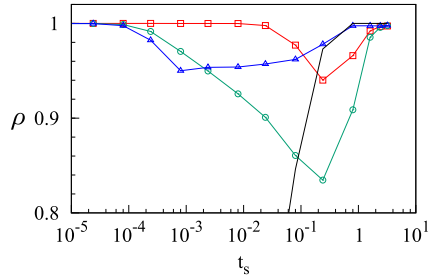

Fig. 2. Dependence of the Pearson product-moment correlation coefficient ρ on the scaled time t_s. The coefficient ρ is calculated between the time-dependent PDF obtained from the Markovian approximation and the numerically obtained time-dependent PDFs when the initial configuration contains no one-spin domains (red squares), one one-spin domain (green circles), two one-spin domains (blue triangles). The solid curves passing through the symbols are for the convenience of the eyes only. Black solid curve without symbols shows the Pearson product-moment correlation coefficient between the time-dependent PDF obtained from the Markovian approximation and the steady-state PDF. Model parameters are $N = 100$, $d = 4$, $\gamma = 0.005$, $\Delta t = 1$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

randomly generated several initial spin configurations, each with $M = 0$, and observed how well the numerically obtained time-dependent PDFs (calculated using 10^5 realizations that start from same initial configuration and contain $5 \cdot 10^5$ points each) correspond to the time-dependent PDF obtained from the Markovian approximation. The dependence of the Pearson product-moment correlation coefficient ρ on the scaled time t_s is shown in Fig. 2. One can see that the differences in initial spin configuration play a significant role for intermediate times $t_s \lesssim 1$ as the Pearson product-moment correlation coefficient becomes smaller. The initial configuration becomes less important as the time-dependent PDF approaches the steady-state PDF.

4. Long-range temporal dependence in the model

Presence of macroscopic fluctuations in the model presented in previous section can enable recovery of $1/f$ noise. To demonstrate this let us consider a quantity equal to the ratio of the number of spins with projection $-\frac{1}{2}$ to the number of spins with projection $+\frac{1}{2}$:

$$y = \frac{N_-}{N_+} = \frac{1-x}{1+x}. \tag{36}$$

The range of possible y values is $[d/(2N-d), (2N-d)/d]$. Using Itô's lemma one may obtain the following SDE for the stochastic variable y

$$\frac{dy}{dt_s} = \frac{1}{4}\left(2 - \frac{\lambda_1}{2} + \frac{\lambda_2}{2y}\right)(1+y)^3 + \frac{1}{2}(1+y)^2\xi(t_s), \tag{37}$$

where

$$\lambda_1 = \frac{d}{2} + 1, \qquad \lambda_2 = \frac{d}{2} - 1. \tag{38}$$

The steady-state PDF of the stochastic variable y, obtained from the Fokker–Planck equation corresponding to the SDE (37), is

$$P_0(y) = \frac{\Gamma(\lambda_1+\lambda_2)}{\Gamma(\lambda_1-1)\Gamma(\lambda_2+1)} \frac{y^{\lambda_2}}{(1+y)^{\lambda_1+\lambda_2}} = \frac{\Gamma(\lambda_1+\lambda_2)}{\Gamma(\lambda_1-1)\Gamma(\lambda_2+1)} \exp_{q_2}\left(-\frac{\lambda_2}{y}\right) \exp_{q_1}(-\lambda_1 y), \tag{39}$$

where

$$q_1 = 1 + 1/\lambda_1, \qquad q_2 = 1 + 1/\lambda_2. \tag{40}$$

Thus the steady-state PDF of y is a q-exponential with q-exponential cut-off at small values of y. The SDE (37) for large $y \gg 1$ coincides with the nonlinear SDE proposed in Refs. [32,33] (with the power-law exponent in the drift term $\eta = 2$). This similarity implies that SDE (37) should generate time series exhibiting power spectral density (PSD) of $S(f) \sim 1/f^\beta$ form, where

$$\beta = 1 + \frac{\lambda_1 - 3}{2(\eta - 1)} = \frac{d}{4}. \tag{41}$$

This expression is valid for $0 < \beta \leqslant 2$. Thus we may expect that the PSD of the ratio N_-/N_+ should be $1/f$ in a wide range of frequencies when $d = 4$. The range of frequencies is limited by the finite number of spins (the ratio N_-/N_+ has a finite maximum possible value $y_{\max} \approx 2N/d$) as well as by the steady-state PDF of y exhibiting power-law behavior with the power-law exponent λ_1 only for $y \gg 1$. According to Ref. [34] the frequencies in the power-law part of the PSD satisfy $\sigma^2 y_{\min}^{2(\eta-1)} \ll 2\pi f_s$, where σ is the coefficient in the noise term. For SDE (37) $y_{\min} = 1$ and $\sigma = \frac{1}{2}$. Going back from the scaled time t_s to the physical time t we get that the PSD has power-law behavior for frequencies

$$\frac{\gamma d}{N^2} \ll 2\pi f. \tag{42}$$

We see that the lowest limiting frequency decreases with increase of the number of spins as $1/N^2$. The width of the frequency region with the power-law behavior of the PSD grows with increasing particle number N.

The power-spectral density of the ratio $y = N_-/N_+$ obtained from numerical simulation of the stochastic evolution of the ring of spins when $d = 4$ is shown in Fig. 3. We see a good agreement of the numerical results with the analytical predictions. The numerical simulation confirms the presence of $1/f$ region in the spectrum of the ratio N_-/N_+. The $1/f$ interval in the PSD in Fig. 3(b) is approximately between $f_{\min} \approx 10^{-9}$ and $f_{\max} \approx 10^{-6}$, the lowest frequency $f_{\min}$ is in agreement with the estimation (42). The width of this frequency interval can be increased by increasing N.

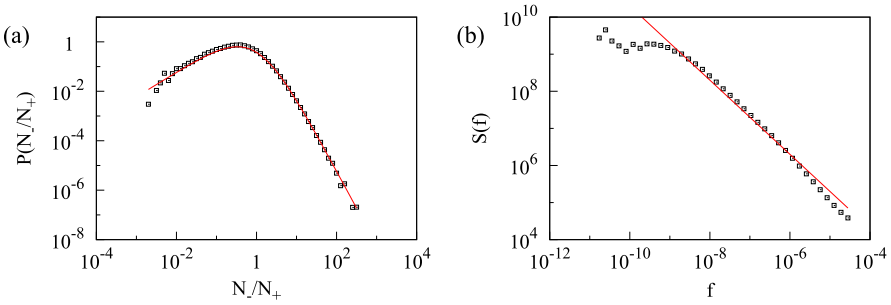

Fig. 3. (a) The steady-state PDF of the ratio N_-/N_+. Solid (red) line corresponds to Eq. (39). (b) The PSD of the ratio N_-/N_+. Solid (red) line shows the slope f^{-1}. Black squares show numerical results obtained using simulation of the stochastic evolution of the ring of spins. Model parameters are $\gamma = 0.005$, $N = 10^3$, $d = 4$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

5. Interacting subsystems

The temporal dynamics in the model of correlated spins, proposed in Section 3, allows us to consider a contact between two such systems. Since the dynamics results in a random walk of the spin flip positions, it is natural to assume that the contact between two systems allows for spin flips to move from one system to the other. If the first system has d_1 spin flips, the second system has d_2 spin flips, then the total number of spin flips in the composite system is $d = d_1 + d_2$. Taking into account the relation $q_{\text{stat}} = 1 - 1/d$ [17], we obtain that the index q_{stat} of the composite system obeys the equation

$$\frac{1}{1-q_{\text{stat}}} = \frac{1}{1-q_{\text{stat},1}} + \frac{1}{1-q_{\text{stat},2}}. \tag{43}$$

Note, that for such composite system the number of allowed microscopic configurations W is not equal to the product $W_1 W_2$, where W_i is the number of allowed microscopic configurations of the isolated i-th system. Since the number of allowed microscopic configurations grows as $W \sim N^d$ [17] and for composite system $N = N_1 + N_2$ with $d = d_1 + d_2$, we have

$$W \sim (W_1^{1-q_{\text{stat},1}} + W_2^{1-q_{\text{stat},2}})^{\frac{1}{1-q_{\text{stat}}}}. \tag{44}$$

This equation generalizes Eq. (3.113) of Ref. [1] for subsystems with different values of q.

The subsystem of larger composite system has fluctuating number of spin flips. Let us calculate the probability distribution of spin flips in the subsystems when this distribution becomes stationary. In this situation we can assume that each microscopic configuration of the composite system has the same probability. The number of ways to partition a ring of N spins into d domains is equal to the number of ways $\binom{N}{d}$ to place d domain boundaries into N possible positions. This number should be multiplied by 2 because there are 2 ways to assign the signs of spin projections to the domains. Thus the number of allowed microscopic configurations of the ring of spins is

$$W = 2\binom{N}{d} \tag{45}$$

Let us consider a subsystem as a spin chain having N_1 spins being a part of a larger ring with N spins. The number of microscopic configurations where the subsystem has d_1 domain boundaries is proportional to the number of ways to place d_1 domain boundaries into $N_1 - 1$ positions in the subsystem multiplied by the number of ways to place remaining $d - d_1$ domain boundaries into remaining $N - N_1 + 1$ positions:

$$W(d_1) = 2\binom{N_1 - 1}{d_1}\binom{N - N_1 + 1}{d - d_1} \tag{46}$$

The probability to have d_1 boundaries in the subsystem is equal to the ratio

$$P(d_1) = \frac{W(d_1)}{W}. \tag{47}$$

Note that the probabilities are normalized, $\sum_{d_1=0}^{d} P(d_1) = 1$. Using Eqs. (45)–(47) for large $N_1 \gg d_1$ and $N \gg d$ we obtain the distribution of the spin flips in the subsystem

$$P(d_1) \approx \binom{d}{d_1}\left(\frac{N_1}{N - N_1}\right)^{d_1}\left(\frac{N - N_1}{N}\right)^{d}. \tag{48}$$

The average number of spin flips in the subsystem calculated using the probabilities (47) is

$$\langle d_1 \rangle = d\frac{N_1}{N}. \tag{49}$$

Thus for the two subsystems in contact we have the equality

$$\frac{\langle d_1 \rangle}{N_1} = \frac{\langle d_2 \rangle}{N_2} \tag{50}$$

when distributions of spin flips in the subsystems become stationary. We can interpret the quantity $\Theta = \langle d_1 \rangle / N_1$ as an effective temperature. An effective temperature associated with the non-extensive statistical mechanics and proportional to the density of vortices in type II superconductors has been introduced in [35,36].

6. Conclusions

We have extended the statistical model proposed in Ref. [17] that exhibits both extensive behavior of generalized entropy for $q < 1$ and a q-Gaussian distribution. Assuming that the stochastic temporal dynamics is due to random walk of domain boundaries we have derived the Fokker–Planck equation (29) and corresponding SDE (30) describing the evolution of the total spin in time. Although distributions from non-extensive statistical mechanics are more often obtained using nonlinear Fokker–Planck equations [1], our model is well described by a linear Fokker–Planck equation. Eq. (29) is a particular case of a known class of Fokker–Planck equations, which have q-Gaussian stationary distributions [30]. The proposed temporal dynamics in the model provides insight on how Fokker–Planck equations with stationary distributions of non-extensive statistical mechanics can arise from microscopic description of the system.

In contrast to the model with uncorrelated spins, presented in Section 2, the dynamics in the proposed model of correlated spins slow down with the increasing number of spins as $1/N^2$. In addition, the fluctuations of the total spin remain macroscopic even in the limit $N \to \infty$, whereas in the model with uncorrelated spins the fluctuations decrease as $1/\sqrt{N}$. These macroscopic fluctuations is one of the reasons for the spectrum of fluctuations of the ratio N_-/N_+ exhibiting $1/f$ behavior in a wide range of frequencies. This range of frequencies increases with increasing number of spins N.

The model presented in this Letter works for $q < 1$. Thus the question remains whether it is possible to modify the model to obtain q-Gaussian distribution with $q > 1$.

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