Evolution of Complex Systems and 1/f Noise:
from Physics to Financial Markets

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Abstract. We introduce the stochastic multiplicative model of time intervals between the events, defining a multiplicative point process and analyze the statistical properties of the signal. Such a model system exhibits power-law spectral density $S(f) \sim 1/f^\beta$, scaled as power of frequency for various values of β between 0.5 and 2. We derive explicit expressions for the power spectrum and other statistics and analyze the model system numerically. The specific interest of our analysis is related with the theoretical modeling of the nonlinear complex systems exhibiting fractal behavior and self-organization.

Introduction

The power spectra of a large variety of different evolutional systems at low frequencies have 1/f behavior. 1/f noise has been observed in condensed matter systems, river discharge, DNA base sequence structure, cellular automata, traffic flow, economics, financial markets and other complex systems with the elements of self-organization (see, e.g., [1-3] and references herein). Considerable parts of such systems are fractal and their statistics exhibit scaling. The universality of 1/f noise suggests that it does not arise as a consequence of particular interaction but it is a characteristic signature of complexity. Computer experiments with various cellular automata and with other nonlinear systems illustrate how this might appear.

There are a number of the self-organized systems. One of the most popular of them is the sandpile model [4]. These nonlinear model systems were introduced also for explanation of 1/f noise as a result of the *self organized criticality*. The great interest of research in this direction is related with the complex behavior which mimics a noise with fractal characteristics. The noise, however, is originated from the nonlinear *deterministic* systems.

From our point of view it is possible to define a *stochastic* model system exhibiting fractal statistics and 1/f noise, as well. Such model system may represent the limiting behavior of the dynamical or deterministic complex systems, explaining the evolution of the complexity into chaotic regime. On the other hand, the stochastic point processes, analyzed in this paper, may be used for description of phenomena that occur as random sequences of events, exhibiting scaling of several statistics [5]. Considerable part of real stochastic sequences of events in physics, biomedicine, geophysics and economics are fractal. The proposed point process model of such systems has an evident physical meaning, because in the low frequency limit the real pulses may be represented as the point events.

The aim of this contribution is an introduction of the multiplicative stochastic model for the time interval between events in stochastic sequence, defining in such a way the multiplicative point process. We adopt the model of 1/f noise based on the Brownian motion of the time interval between subsequent pulses proposed in Refs. [3,6,7] introducing the stochastic multiplicative model of the interevent time [8]. The model exhibits the first order and the second order power-law statistics and serves as the theoretical description of the empirical trading activity in the financial markets [9].

Our specific interest is an analysis of the relation between the origin of the power-law distributions and the power-law correlations. Obviously, the multiplicative point process can be

useful for the modeling of a wide variety of natural systems as well as of the processes in economics and finance. We expect that the multiplicative point process can serve as a limiting theoretical approach in the analysis of the nonlinear complex systems with the elements of self-organization and for analysis of transition to the chaotic behavior.

Multiplicative Point Process

We consider a signal $I(t)$ as a sequence of the random correlated pulses

$$I(t) = \sum_k a_k \delta(t - t_k) \tag{1}$$

where a_k is a contribution to the signal of one pulse at the time moment t_k, for example, a contribution of one transaction to the financial data. When a_k is a constant, the process (1) is completely defined by the set of events $\{t_k\}$ or equivalently by the set of interevent intervals $\{\tau_k = t_{k+1} - t_k\}$. Kaulakys and Meškauskas [3,6,7] showed analytically that the relatively slow Brownian fluctuations of the interevent time τ_k exhibited $1/f$ fluctuations of the signal $I(t)$.

Power spectral density of the signal (1) can be written as

$$S(f) = \lim_{T \to \infty} \left\langle \frac{2a^2}{T} \sum_{k=k_{\min}}^{k_{\max}} \sum_{q=k_{\min}-k}^{k_{\max}-k} \exp(-i2\pi f \Delta(k;q)) \right\rangle \tag{2}$$

where T is the observation time, $\Delta(k;q) = t_{k+q} - t_k$ is the difference of pulses occurrence times t_{k+q} and t_k, a denotes expectation of a_k, while $k_{\min}$ and $k_{\max}$ are minimal and maximal values of index k in the time interval of observation T.

We will study the multiplicative processes defined by the stochastic iterative equation

$$\tau_{k+1} = \tau_k + \gamma \tau_k^{2\mu-1} + \tau_k^{\mu} \sigma \varepsilon_k. \tag{3}$$

Here the interevent time τ_k fluctuates due to the external random perturbation by a sequence of uncorrelated normally distributed random variable ε_k with a zero expectation and unit variance, σ denotes the standard deviation of the white noise and $\gamma = 1$ is a damping constant of the signal. The diffusion described by Eq. (3) has to be restricted in some time interval $\tau_{\min} < \tau < \tau_{\max}$.

Pure multiplicativity corresponds to the parameter $\mu = 1$. Nevertheless, other values of μ can produce power laws, as well, and the explicit expressions can be derived without the loss of generality. The iterative relation (3) can be rewritten as a continuous Langevine stochastic differential equation in k space

$$\frac{d\tau}{dk} = \gamma \tau_k^{2\mu-1} + \tau_k^{\mu} \sigma \xi_k \tag{4}$$

where $\langle \xi_k \xi_{k'} \rangle = \delta(k - k')$. The stationary solution of the corresponding Fokker-Plank equation with a zero flow gives the long time probability distribution of τ in the space k

$$P_k(\tau) = C_\tau \tau^\alpha \tag{5}$$

where C_τ has to be defined from the normalization $\int_{\tau\min}^{\tau\max} P(\tau)d\tau = 1$ and $\alpha = \dfrac{2\gamma}{\sigma^2} - 2\mu$.

Power Spectral Density and Counting Statistics

The power spectral density is a well-established measure of long-time correlations and is widely used in stochastic systems. For the normal distribution of $\Delta(k;q)$ Eq. (2) takes the form

$$S(f) = \lim_{T\to\infty} \frac{2a^2}{T} \sum_{k,q} \exp\{i2\pi f \langle\Delta(k;q)\rangle - 2\pi^2 f^2 \sigma_\Delta^2(k;q)\} \tag{6}$$

where $\Delta(k;q)$ can be expressed from the solution of the multiplicative stochastic equation (4),

$$\Delta(k;q) = \sum_{i=1}^{q} \tau_k \exp((\gamma - \frac{1}{2}\sigma^2)\tau^{2\mu-2}i + \sum_{j=1}^{i} \sigma\tau^{\mu-1}\varepsilon_j). \tag{7}$$

Averaging over the normal distributions of ε_j yields the explicit expressions for the mean $\langle\Delta(k;q)\rangle$ and variance $\sigma_\Delta^2(k;q)$,

$$\langle\Delta(k;q)\rangle = \tau_k q + \frac{\gamma}{2}\tau_k^{2\mu-1}q^2 + o(\gamma^2), \tag{8}$$

$$\sigma_\Delta^2(k;q) = \tau_k^{2\mu} \frac{\sigma^2}{3} q^3. \tag{9}$$

In the low frequency limit Eqs. (6) and (8) yield the power spectral density

$$S_\mu(f) = \frac{2C_\tau a^2}{\sqrt{\pi}\bar\tau(3-2\mu)f} (\frac{\gamma}{\pi f})^{\frac{\alpha}{3-2\mu}} \mathrm{Re} \int_{x\min}^{x\max} \exp(-i(x-\frac{\pi}{4}))erfc(\sqrt{-ix})x^{\frac{\alpha}{3-2\mu}-\frac{1}{2}}dx. \tag{10}$$

Here we introduce the scaled variable $x = \dfrac{\pi f}{\gamma}\tau^{3-2\mu}$ and $\bar\tau$ is the expectation of τ_k. For $x_{\min} \to 0$ and $x_{\max} \to \infty$ Eq. (10) yields the explicit form of the power spectrum

$$S_\mu(f) = \frac{C_\tau a^2}{\sqrt{\pi}\bar\tau(3-2\mu)f} (\frac{\gamma}{\pi f})^{\frac{\alpha}{3-2\mu}} \frac{\Gamma(\frac{1}{2}+\frac{\alpha}{3-2\mu})}{\cos(\frac{\pi\alpha}{2(3-2\mu)})}. \tag{11}$$

Eq. (11) proves that the multiplicative point process exhibits a general model of signals with the power spectral density $S(f) \sim f^{-\beta}$. The scaling exponent is

$$\beta = 1 + \frac{2\gamma/\sigma^2 - 2\mu}{3-2\mu}. \tag{12}$$

Numerical calculations confirm the validity of Eqs. (10), (11) and (12).

In the Fig. 1 and Fig. 2 we provide results of numerical calculations for the various values of parameters. We suppose that this stochastic model with the parameters resulting in β; 1 can be adopted for a wide variety of real systems.
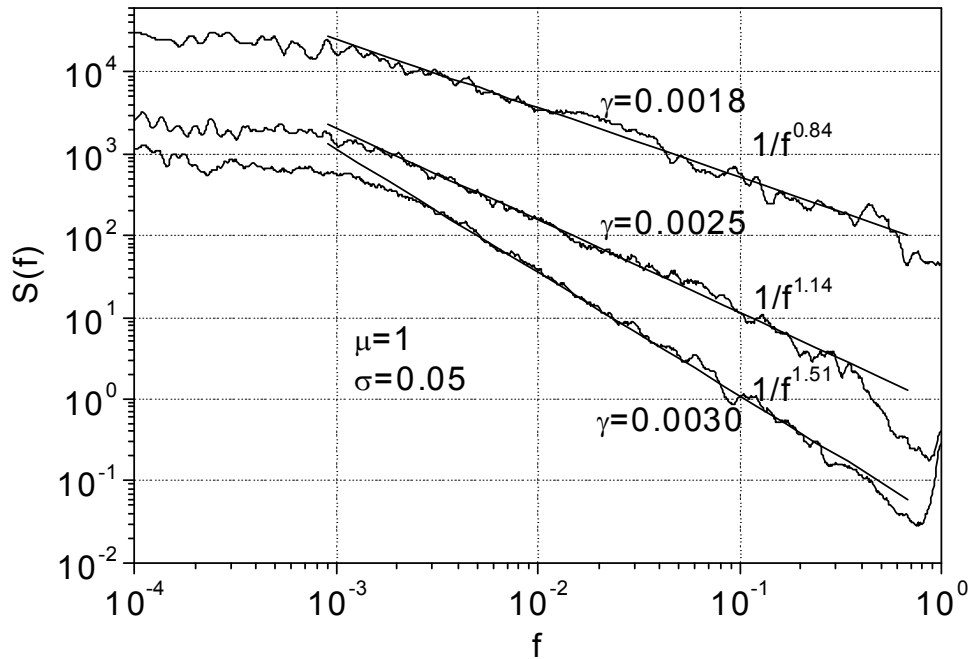

Fig. 1. Power spectral density vs frequency of the signal generated by the Eqs. (1), (2) and (3) with the parameters $\mu = 1$, $\sigma = 0.05$ and different relaxations of the signal γ. We restrict the diffusion of the interevent time in the interval $\tau_{min} = 10^{-6}$, $\tau_{max} = 1$ with the reflective boundary condition at τ_{min} and transition to the white noise, $\tau_{k+1} = \tau_{max} + \sigma\varepsilon_k$, for $\tau_k > \tau_{max}$.
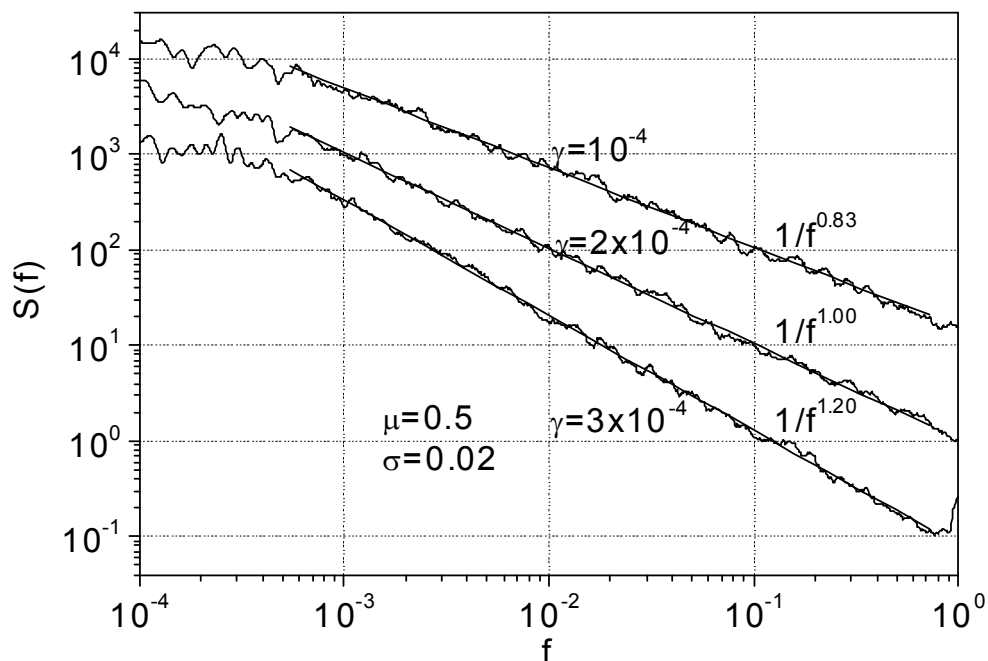

Fig. 2. The same as in Fig. 1 but for $\mu = 0.5$, $\sigma = 0.02$ and different parameters γ.

Let us assume that $a \equiv 1$ and the signal $I(t)$ counts the transactions in financial markets. Then the number of transactions in the selected time window τ_d defined as $N(t) = \int_{t}^{t+\tau_d} I(t)dt$ measures the trading activity. For the pure multiplicative model, $\mu = 1$, Eqs. (7) and (8) define the relation between N and τ. After substitution $\tau_k \to \tau, q \to N$ and $\langle \Delta(k;q) \rangle \to \tau_d$ we get

$$\tau N + \frac{\gamma}{2} \tau N^2 = \tau_d .$$ (13)

This relationship may be used for definition of the probability density function of N in a real time from the relation $P_t(N)dN = P_t(\tau)d\tau$,

$$P_t(N) = \frac{C_t \tau_d^{2+\alpha}(1+\gamma N)}{N^{3+\alpha}(1+\frac{\gamma}{2}N)^{3+\alpha}} ; \quad \left\{ \begin{array}{ll} \frac{C_t \tau_d^{2+\alpha}}{N^{3+\alpha}}, & N = \gamma^{-1}, \\ \frac{C_t 2^{3+\alpha}(\tau_d/\gamma)^{2+\alpha}}{N^{5+2\alpha}}, & N ? \gamma^{-1} \end{array} \right\} .$$ (14)

In the case of pure multiplicativity, $\mu = 1$, the model has only one parameter, $2\gamma/\sigma^2$, defining the scaling of the power spectral density, the power-law distributions of interevent time and the counting number N. The model proposed with the adjusted parameter $2\gamma/\sigma^2$ nicely describes the empirical power spectral density and the exponent of power-law distribution of the trading activity N in the financial markets [10].

Conclusions

We introduce a multiplicative stochastic model for the time intervals between events of the process seeking to define the variety of self-affine time series exhibiting the power-law spectra of different slopes. Such a model of time series has only a few parameters. However, it defines the statistical properties of the system, i.e., the power-law behavior of the distribution function and the scaled power spectral density of the signal. Such power-law phenomena are observable in a large variety of processes, from earthquakes to the financial time series. The proposed model relates and connects the power-law correlations and the power-law distribution of the signal intensity into the consistent theoretical approach. The generated time series are fractal since they exhibit jointly the power-law probability distribution and the power-law autocorrelation of the signal. The ability of the model to simulate 1/f noise as well as to reproduce signals with the values of power spectral density slope between 0.5 and 2 promises wide applications of the model.

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